ON THE DUALISABILITY OF FINITE \{0,1\}-VALUED UNARY ALGEBRAS WITH ZERO

by

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B.Sc., University of Northern British Columbia, 2010

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS

UNIVERSITY OF NORTHERN BRITISH COLUMBIA

June 2014

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Abstract

This thesis provides a few results regarding the natural dualisability of certain \( \{0,1\}\)-valued unary algebras with zero. We use pp-formulae to develop a sufficient criterion for non-dualisability of such algebras. With this criterion, we show that \( \{0,1\}\)-valued unary algebras with zero with unique rows whose rows form an order ideal (with respect to the lattice order \( \{0,1\}^n \) under \( 0 < 1 \)) are not dualisable if its rows do not form an lattice order. For the case where the rows do form a lattice, we use the Interpolation Condition to show that the algebra is dualisable. The last result of this thesis provides another sufficient criterion for non-dualisability by looking at two-term reducts.
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Dedication

To Cathy Stevenson.
Acknowledgement

First and foremost, I want to thank my supervisor Jennifer Hyndman. I would like to thank David Casperson for providing helpful discussion, and valuable criticism during the drafting phase of my thesis. I thank Mark Shegelski for going out of his way to help me satisfy my course requirements. I thank Jesse Mason, Jane Pitkethly, and Allan Kranz for their help with \LaTeX. I appreciate Sarah Pickett, Amelia Garcia, and my family Grant Schaan, Katie Molloy, and Adam Schaan for their support throughout. Finally, I thank the University of Northern British Columbia and the Natural Sciences and Engineering Research Council of Canada for their financial support.
Chapter 1

Introduction

1.1 Statement of the Problem and Summary of Results

This thesis looks at the natural dualisability of \(\{0,1\}\)-valued unary algebras with zero. When beginning this thesis, the question we sought to answer was which \(\{0,1\}\)-valued unary algebras with zero whose rows form an order ideal are dualisable? In researching this question, a pattern emerged which is formalized in Chapter 3. The main result from Chapter 3 is Theorem 8 which uses pp-formulæ to provide a sufficient condition for non-dualisability of \(\{0,1\}\)-valued unary algebras with zero. The remaining results of this thesis regarding non-dualisability use Theorem 8 in their proofs. Theorem 11 is the main result of Chapter 4. It states that a \(\{0,1\}\)-valued unary algebra with zero with unique rows whose rows form an order ideal is dualisable if and only if its rows form a lattice order. This answers our original question when the rows of the algebra are uniquely witnessed. Chapter 5 provides Theorem 19, which is a simple test for non-dualisability of \(\{0,1\}\)-valued unary algebras with zero with unique rows. This thesis concludes with a few examples which demonstrate the scopes and limitations of Theorems 8, 11, and 19. These limitations suggest topics for future research to extend our results.
1.2 Related Work

The goal of this section is to provide some perspective of how the results of this thesis fit within research that has been conducted on unary algebras, and on dualisability. We also mention some results on finite bases of equations or quasi-equations, as these results appear to have some relationship with dualisability.

In [10], Davey et al. classify the dualisability of finite graph algebras. The relevance of this result to this thesis lies more in the development of the paper than the results themselves. The paper uses as a foundation a characterization of Baker, McNulty, and Werner ([1]) of the finite graph algebras with finite bases for their equations. The parallel to our work is that the topic of this thesis is inspired by a categorization of the \( \{0,1\} \)-valued unary algebras with 0 with finite bases for their quasi-equations found in [4]. Although dualisability and (quasi-)equational theory at first appear to be entirely separate concepts, there are some unusual and not entirely understood connections between them. A result of [10] is that a finite graph algebra is dualisable if and only if it has a finite basis for its equations. A similar question is under what conditions does such a strong connection exists between dualisability and finite bases for quasi-equations. Hyndman and Pitkethly ([11]) find several examples of three-element unary algebras with finite bases of their quasi-equations that are not dualisable. In the same paper, however, the authors establish that a three-element unary algebra that is dualisable is strongly dualisable if and only if it has a finite basis for its quasi-equations. These kinds of results suggest that developing a better understanding of natural dualities will lead to better understanding of other algebraic properties of interest.

A great deal of the research that has been done on dualisability of unary algebras has been focused on strong dualities. Willard defined the rank of an algebra which he used to provide a sufficient condition for a finite dualisable algebra to be strongly dualisable (Theorem 4.1 [21]).
Lampe, McNulty, and Willard defined enough algebraic operations to simplify the idea of rank, which they prove is still sufficient for a finite dualisable algebra to be strongly dualisable (Theorem 4.3 [14]). In [12], Hyndman shows that finite unary algebras for which particular relations can be pp-defined do not have enough algebraic operations. In this thesis, Theorem 8 (upon which the other results of this thesis are based) demonstrates that pp-formulae can also be used to determine non-dualisability. Pitkethly and Davey provide another concept based on rank called height, which is both necessary and sufficient for a finite dualisable algebra to be strongly dualisable (A.4.6 Theorem [16]).

Another question to ask is why we should be interested in such a specific collection of algebras as \{0, 1\}-valued unary algebras with zero. The following quote from [10] (p. 149) elucidates: “To understand the connection between dualizability and the finite basis property, it seems necessary to consider more pathological finite algebras.” There has been some history of finding interesting dualisability results by looking at such “pathological” algebras, in particular those that are unary. The first algebra known to be dualisable but not fully dualisable by any set of algebraic relations and (partial) operations was a three-element unary algebra found by Hyndman and Willard ([13]). Beveridge ([2]) used rank to find an infinite collection of unary algebras (which includes the aforementioned three-element one) which are dualisable but not strongly dualisable. Unary algebras provide a context that is broad enough to find such counter-examples, while still being “simple” enough to find some general results. There are two natural ways of dividing unary algebras in order to provide general results: place a restriction on the size of the algebra, or on the number or range of operations of the algebra. Clark, Davey, and Pitkethly took the former method, and fully categorized the dualisability of three-element unary algebras ([15]), as well as the strong dualisability of these algebras ([17]). Casperson et al. ([4]) decided to use the latter method by looking at \{0, 1\}-valued unary algebras with 0. They provide the following classification:
**Theorem 21.** ([4]) If $\mathbf{M}$ is a $\{0,1\}$-valued unary algebra with 0, then one of the following holds:

1. the $\leq$ relation on $\{0,1\}$ can be positive primitively defined;
2. the graph of addition modulo 2 on $\{0,1\}$ can be positive primitively defined;
3. the rows of $\mathbf{M}$ form an order ideal.

In the first two cases, there is no finite basis for the quasi-equations, and in the last case, there is a finite basis for the quasi-equations.

The third case of this theorem provided the inspiration for the topic of this thesis; in Theorem 11, we categorize the dualisability of the algebras of the third case when the rows of the algebra are unique.

We close this chapter by mentioning a significant recent result. Pitkethly proved that for each dualisable finite unary algebra, there is some finite $n$ such that the algebra can be dualised by a structure whose algebraic relations and operations (both total and partial) are all of arity less than $n$ (Theorem 2.4 [18]). The problem of whether or not such an $n$ exists for a particular collection of algebras is known as The Finite Type problem, and the answer is not known in general when we consider algebras of finite type that are not unary, although a negative solution has been found by Pitkethly for finite algebras of infinite type ([19]). An interesting corollary of Pitkethly’s result in [18] is a technique for proving non-dualisability of finite unary algebras: if one can prove that relations of any finite bounded arity are not sufficient to dualise an algebra, then the algebra is not dualisable.
Chapter 2

Background

In this chapter, we look at the background material in algebra, topology, and natural duality theory that are used throughout this thesis.

A few comments on notation: when illustrating certain concepts, we may be dealing with some arbitrary set \( \{a_i\}_{i \in I} \). Often, we are not particularly worried about the indexing set \( I \) (which, in the most general setting, could be finite, countably or uncountably infinite, or empty), and in these cases, we may write \( \{a_i\} \) instead of \( \{a_i\}_{i \in I} \). We include 0 in the natural numbers \( N \), which we sometimes write as \( \omega \). We reserve capital letters in the middle of the Latin alphabet (such as \( M \) and \( N \)) for the underlying sets of our "base structures". Capital letters at the beginning of the alphabet (such as \( A \) and \( B \)) we use for either arbitrary sets or universes of algebras built from the base structures. Letters at the end of the alphabet (such as \( X \) and \( Y \)) are used for underlying sets of topological structures built from the base structures. Boldfaced letters underlined with a tilde (such as \( \tilde{X} \)) indicate topological structures, while boldfaced and underlined letters (such as \( \tilde{M} \)) indicate algebras.

Much of this chapter is borrowed from a graduate paper written by the author for Math 699:
Topology with permission from both Professor Sam Walters (the professor of the course) and Jennifer Hyndman (the author’s graduate supervisor).

2.1 Algebra

This material comes from [3] and [15] with notational changes as in [6] to be consistent with standard notation in natural duality theory.

We start by establishing the contexts in which natural duality theory occurs. The first is that of algebra. To precisely define what we mean by an "algebra," we start with some basic concepts. A language or type of algebras is a set \( \mathcal{F} \) (whose elements we refer to as function symbols) such that each \( f \) in \( \mathcal{F} \) has associated with it a non-negative integer \( n \) called the arity of \( f \). By an \( n \)-ary operation on a set \( A \), we mean a function from \( A^n \) to \( A \). A finitary operation is an \( n \)-ary operation for some finite \( n \). The projection operations or projections on a set \( A \) are the maps \( \pi_i \) (\( 1 \leq i \leq n \)) from \( A^n \) to \( A \) defined by \( \pi_i(a_1, a_2, ..., a_n) = a_i \). In particular when \( n = 1 \), we call \( \pi_1 \) the identity function, and denote it \( id \). Note that when \( n \) is not given, it is implied from context.

Given a language \( \mathcal{F} \), an algebra \( A \) of type \( \mathcal{F} \) is an ordered pair \( (A;F) \) where \( A \) is a non-empty set called the universe (or underlying set) of \( A \), and \( F \) is a collection of finitary operations on \( A \) such that to each \( n \)-ary function symbol \( f \) in \( \mathcal{F} \) there is an \( n \)-ary operation \( f^A \) on \( A \). Each \( f^A \) is called a fundamental or basic operation of \( A \). A finite composition of fundamental operations together with projection operations is called a term operation or term function. We write \( f \) instead of \( f^A \), except when it may cause ambiguity: if we were working with both \( A = \mathbb{Z}_3 \) and \( B = \mathbb{Z}_5 \), we would want to distinguish between \( +^A \) (addition modulo 3) and \( +^B \) (addition modulo 5).

For \( B \subseteq A \) and \( f \) a term operation of \( A \), we write \( f|_B \) or, when there is no ambiguity, simply \( f \)
to denote the restriction of $f$ to $B$. If $F = \{f_1, f_2, \ldots, f_k\}$ is finite, we say the algebra is of finite type and we typically write $A = \langle A; f_1, f_2, \ldots, f_k \rangle$ instead of $A = \langle A; F \rangle$. All of the algebras considered in this thesis are of finite type. The clone of an algebra $A$, denoted $\text{clone}(A)$, consists of all term operations of $A$. That is, the clone of $A$ is the smallest set of functions closed under compositions and containing all projections and basic operations. In particular, the collection of unary (i.e. 1-ary) term operations on $A$ is called the unary clone of $A$ and is denoted by $\text{clone}_u(A)$. If the universe of $A$ is finite, we say that $A$ is a finite algebra.

The distinction between a language and a set of fundamental operations of an algebra is somewhat subtle. Homonyms provide an apt analogy: the word “bark” on its own is ambiguous, and hence lacks any tangible meaning. From context we could determine whether the word was referring to the bark of a tree, or the bark of a dog. A function symbol $f$ from a language has no context, whereas the fundamental operation $f^A$ provides the algebra $A$ as the context. Take for example $\mathbb{Z}_3$ and $\mathbb{Z}_5$ from above: the function symbol $+$ lacks a concrete interpretation without the context of the algebra.

We can look at how algebras relate to one another. A function $\varphi$ between algebras $A$ and $B$ of the same type $\mathcal{F}$ is a homomorphism if it is operation preserving—that is, for each $n$-ary $f \in \mathcal{F}$ and $a_1, \ldots, a_n \in A$, we have

$$\varphi (f^A(a_1, \ldots, a_n)) = f^B(\varphi(a_1), \ldots, \varphi(a_n)).$$

In other words, we obtain the same result whether we apply the term operation followed by the homomorphism, or the homomorphism followed by the term operation. If $\varphi$ is also bijective, then it is an isomorphism. An isomorphism can be thought of as a relabelling that does not alter the structure. An embedding is a homomorphism that is injective. If there is an embedding $\varphi : B \to A$, we say that we can embed $B$ into $A$. If $B$ can be embedded into $A$, there is an isomorphic copy of
If $A$ and $B$ are two algebras of the same type $\mathcal{F}$ such that $B \subseteq A$ and $f^B$ is the restriction of $f^A$ to $B$ for each $f \in \mathcal{F}$, then $B$ is a subalgebra of $A$. An equivalent definition is if $B$ is a non-empty subset of $A$ such that for each $f \in \mathcal{F}$ and $b \in B$, we have $f^A(b) \in B$, then $B = (B, F|_B)$ (where $F|_B$ is the set consisting of the restrictions to $B$ of the fundamental operations of $A$) is a subalgebra of $A$. A subuniverse of $A$ is a subset of $A$ that is closed with respect to the operations of $A$. Note that $\emptyset$ is a subuniverse of any algebra that does not have nullary operations, but $\emptyset$ is never the universe of a subalgebra. For example, consider the integers under addition $\mathbb{Z} = (\mathbb{Z}, +)$. As adding any two even integers together results in an even integer, the even integers $\mathbb{Z}_e$ under addition form a subalgebra of $\mathbb{Z}$, and $\mathbb{Z}_e$ is a subuniverse of $\mathbb{Z}$. Also, the empty set is a subuniverse of $\mathbb{Z}$, but $(\emptyset, +)$ is not a subalgebra.

The Cartesian product of a collection of sets $\{A_i\}_{i \in I}$ is the set

$$\prod_{i \in I} A_i = \{(a_i)_{i \in I} \mid a_i \in A_i\}.$$ When there is no ambiguity about the indexing set, we often write the Cartesian product as $\prod A_i$.

In the case where all of the sets are equal, i.e. $A_i = A$ for every $i$, we write $A^I$ for the Cartesian product. Given a collection of algebras $\{A_i\}_{i \in I}$ of the same type $\mathcal{F}$, we define the direct product

$$\prod_{i \in I} A_i$$

to be the algebra whose universe is $\prod A_i$ and such that for each $n$-ary $f \in \mathcal{F}$ and $a_1, ..., a_n \in \prod A_i$, we have

$$f^{\prod A_i}(a_1, ..., a_n)(i) = f^{A_i}(a_1(i), ..., a_n(i))$$

for each $i \in I$. In other words, we define the fundamental operations coordinate-wise. Because the lifting of $f^{A_i}$ to $\prod A_i$ is coordinatewise, we denote both simply by $f$. 

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The Cartesian product $A^J$ can be interpreted two ways. One is to think of $A^J$ as consisting of all "sequences" indexed by $J$ of elements of $A$, much like tuples (the word "sequences" is used loosely here, as the indexing set $J$ may not be countable). The second is to think of $A^J$ as consisting of all functions from $J$ to $A$. For example, if $J = \{1, 2, 3\}$ and $A = \{a, b, c\}$, a typical element in the first interpretation of $A^J$ would be $(a,a,b)$. This same element in the second interpretation would be a function $g : J \to A$ given by $g(1) = a$, $g(2) = a$, and $g(3) = b$. Hence when we write $x(i)$ to denote the $i^{th}$ coordinate of $x \in A^J$ in the first interpretation, the notation is consistent with considering $x$ as the function in the second interpretation. Each interpretation has advantages and disadvantages, and both are used in this work. In either case, the fundamental operations of the algebra $A^J$ are defined canonically as in the previous paragraph.

A relation $R$ on a set $A$ is a subset of $A^n$ for some finite non-zero $n$. The value of $n$ is called the arity of the relation, and we call $R$ an $n$-ary relation. By a relation on an algebra $A$, we mean a relation on $A$.

Using isomorphisms, subalgebras, and direct products, we develop a collection of algebras that provide half the context for natural duality theory. The quasivariety of an algebra $A$ is the collection of all isomorphic images of subalgebras of direct products of $A$. Alternatively, the quasivariety of $A$ is the smallest collection of algebras containing $A$ that is closed under isomorphic copies, subalgebras, and direct products. That these two definitions are equivalent is an exercise in [3] (Exercise 1, p. 68). We denote the quasivariety of the algebra $A$ as $\text{ISP}(A)$.

### 2.2 Posets, Lattices, and Order Ideals

The algebraic structures addressed in this section provide a "meta-structure" for the algebras in which we are interested. This idea is elaborated upon in the following section on unary algebras.
A partial order $\leq$ on a set $A$ is a binary (that is, 2-ary) relation that is reflexive ($a \leq a$ for every $a \in A$), transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$), and antisymmetric (if $a \leq b$ and $b \leq a$, then $a = b$).

The canonical “less than or equal to” on the real numbers is a partial order (hence the use of the $\leq$ symbol). Another example is $a \leq_d b$ iff $b$ is divisible by $a$ on the set of positive integers: certainly $a$ divides $a$ for any positive integer $a$ (reflexivity); if $a$ divides $b$ and $b$ divides $c$, then $a$ divides $c$ (transitivity); and if $a$ divides $b$ and $b$ divides $a$, then necessarily $a = b$ (antisymmetry). Note that, for example, 2 does not divide 3 nor vice-versa, so that $2 \not\leq_d 3$ and $3 \not\leq_d 2$. The divisibility example is a better illustration of partial orders in general than less than or equal to, as it may be the case in a partial order that for a particular $a$ and $b$ in $A$ neither $a \leq b$ nor $b \leq a$ and we say that $a$ and $b$ are incomparable. If $\leq$ forms a partial order on $A$, we say that $\langle A; \leq \rangle$ is a partially ordered set or more commonly a poset.

A nice visual way to represent any poset (including lattices and order ideals) is with a Hasse diagram. Given a poset $\langle A; \leq \rangle$, we represent each element of $A$ as a point. If $a < b$ and there is no $c$ strictly between $a$ and $b$, we draw $a$ below $b$ and draw a line between them. If two elements $x$ and $y$ are incomparable, no line is drawn between them. In practice, it is impossible to fully draw an infinite poset, but we may draw a finite portion of it to illustrate it. Figures 2.1 and 2.2 provide examples of finite posets.

Given a poset $\langle A; \leq \rangle$, we define the binary operations $\wedge$ (called meet) and $\vee$ (called join) as follows: for any pair of elements $a$ and $b$ in $A$, $a \wedge b$ is the largest element in $A$ (using $\leq$) less than or equal to both $a$ and $b$ if it exists, and $a \vee b$ is the smallest element in $A$ (using $\leq$) greater than or equal to both $a$ and $b$ if it exists. Using our above example of divisibility, $4 \wedge 6 = 2$ as 2 divides both 4 and 6 and if $x$ divides both 4 and 6, then $x$ divides 2 (hence 2 is the largest element using $\leq_d$ that divides both 4 and 6). Similarly, $4 \vee 6 = 12$ as 12 is the smallest integer (again using $\leq_d$) divisible by both 4 and 6. It is important to note that for some poset $\langle A; \leq \rangle$, the operations $\wedge$
and ∨ may not be defined for all pairs of elements. Consider for example divisibility on the set $A = \{1, 2, 3\}$: there is no element of $A$ that both 2 and 3 divide, so $2 \lor 3$ is not defined. If it is the case that both $\land$ and $\lor$ are defined for every pair of elements in $A$, then we say $(A; \land, \lor)$ is a lattice. In this case, we also say that $(A; \leq)$ is a lattice order.

For a poset $(A; \leq)$ and $S$ a subset of $A$, we define $\inf(S)$ to be the largest element in $A$ less than or equal to every element in $S$ (if it exists), and $\sup(S)$ to be the smallest element in $A$ greater than or equal to every element in $S$ (if it exists). The operations $\inf$ and $\sup$ are called the infimum and supremum respectively.

Let $(A; \leq)$ be a poset, and let $B \subseteq A$. If for every $a$ in $A$ such that $a \leq b$ for some $b$ in $B$, we have $a$ in $B$, then $(B; \leq)$ is an order ideal. In less formal language, an order ideal is a downward closed subset of a set with a partial order. Returning to divisibility on the positive integers, if $B$ is an order ideal containing the elements 15 and 2, $B$ would also have to include the elements 1, 3, and 5 (as these are the elements $\leq_d 15$ or $\leq_d 2$). Thus $(\{1, 2, 3, 5, 15\}; \leq_d)$ is an order ideal. Figure 2.2 illustrates this order ideal. Notice that order ideals are closed under $\land$. Indeed, by definition $a \land b \leq a$ (and $\leq b$), so because $a$ is in the order ideal, so too must be $a \land b$. Although closure under
\( \wedge \) is necessary for an order ideal, it is not sufficient: using \( \leq_d \), we have \( 3 \wedge 3 = 3 \), \( 3 \wedge 6 = 3 \), and \( 6 \wedge 6 = 6 \), so \( \{3, 6\} \) is closed under \( \wedge \). However, \( 2 \leq_d 6 \) and \( 2 \notin \{3, 6\} \), so \( \{\{3, 6\}; \leq_d\} \) is not an order ideal.

### 2.3 Unary Algebras

In this work, we categorize the dualisability of a particular class of finite unary algebras of finite type. A **unary algebra** is an algebra \( P = \langle P; F \rangle \) such that every operation \( f \) in \( F \) is **unary** (that is, 1-ary). We see later in Lemma 1 that the algebras considered have unary clones that differ from the fundamental operations by at most the identity and the constant 0-valued function. The unary clone of a finite unary algebra can be completely described using a table as in Figure 2.3. The left-most column contains the elements of the universe. The top-most row consists of the entire unary clone of the algebra except the identity function. The entries in the table correspond to the behaviour of the term operation on that particular element: in Figure 2.3 for example, the entry in the \((0, f_1)\) position specifies that \( f_1(0) = 2 \). We can also represent unary algebras visually, as in Figure 2.4. Each element is drawn as a point, and each fundamental operation is associated a different style of line (such as solid or dashed). Arrows are drawn to denote the behaviour of the fundamental operations: for example, in Figure 2.4, the operation \( f_1 \) is represented by solid lines and a solid line with an arrow is drawn from the point 0 to the point 2 to denote \( f_1(0) = 2 \). Loops are used to denote \( f(x) = x \), such as the solid loop around the point 2 to denote \( f_1(2) = 2 \). The term operation \( f_3 \) is omitted from Figure 2.4 as it can be written as the composition \( f_3 = f_2 \circ f_1 \).

Given a unary algebra \( P \) of finite type, we define \( F_0 = \{f_1, f_2, \ldots, f_n\} \) to be the set of all non-constant, non-identity fundamental operations of \( P \), and \( F_c \) to be the set of all constant-valued fundamental operations of \( P \). Note that \( P = \langle P; F_c \cup F_0 \cup \{id\} \rangle \). Given an element \( a \) of a unary
algebra $P$, we define the tuple $\text{row}(a) = (f_1(a), f_2(a), \ldots, f_n(a))$. By combining all the rows of the algebra together, we define the relation $\text{Rows}(P) = \{\text{row}(p) \mid p \in P\}$. Note that the definition of $\text{row}(a)$ and hence the definition of $\text{Rows}(P)$ are dependent on the ordering of the $f_i$ in $F_0$. When not stated explicitly, the ordering on $F_0$ is arbitrary but fixed. We refer to a specific row $\text{row}(a)$ in $\text{Rows}(P)$ as the row witnessed by $a$. Note that for any $a$ in $P$, $\text{row}(a)$ is an element of $P^n$, and so $\text{Rows}(P)$ is a subset of $P^n$. Using Figure 2.3 as an example, $f_1$ is the constant 2-valued function, so we have $\text{Rows}(P) = \{(2,1), (1,2), (2,2)\}$ where $(2,1)$ is witnessed by 0, $(1,2)$ is witnessed by 1, and $(2,2)$ is witnessed by 2. In this case, each entry in $\text{Rows}(P)$ has exactly one witness, and thus we say that $\text{Rows}(P)$ is uniquely witnessed.

In particular, the finite unary algebras analysed in this work are those such that the range of all basic term operations is a two element set $\{0, 1\}$ such that $\{0\}$ is a one-element subuniverse and such that there is a constant valued function with image 0. We call such algebras $\{0, 1\}$-valued unary algebras with zero. In terms of the table representation, being $\{0, 1\}$-valued means the entries of the table are all either zeros or ones. Having $\{0\}$ as a one-element subuniverse means that row$(0)$ contains only zeros, and having 0 as a constant valued function means there is a column of zeros. Because $\{0, 1\}$-valued unary algebras with zero always have the constant 0 valued function, we occasionally omit it when writing out the tables for $\{0, 1\}$-valued unary algebras with zero.
Figure 2.5 provides an example of a \( \{0,1\} \)-valued unary algebra with zero. Note that for any \( \{0,1\} \)-valued unary algebra with zero \( M \), \( \text{Rows}(M) \) is contained in \( \{0,1\}^n \). Since \( \{0,1\}^n \) forms a lattice under \( 0 < 1 \), we may consider the inherited poset structure on \( \text{Rows}(M) \). This treatment is used throughout this thesis.

There are several advantages to working with \( \{0,1\} \)-valued unary algebras with zero. One advantage is that it is easy to determine the unary clone as demonstrated by this lemma:

**Lemma 1.** If \( M = (M;F) \) is a \( \{0,1\} \)-valued unary algebra with zero, then \( \text{clone}_u(M) = F \cup \{f_0, \text{id}\} \) where \( f_0 \) is the constant 0-valued term function.

**Proof.** It suffices to show that given \( f \) and \( g \) in \( F \), either \( f \circ g \in F \) or \( f \circ g = f_0 \). If either \( f \) or \( g \) is the identity term, then either \( f \circ g = g \) or \( f \circ g = f \) which are both in \( F \). If \( f(1) = 1 \), then for any \( m \in M \), we have \( f(g(m)) = f(1) = 1 \) if \( g(m) = 1 \) and \( f(g(m)) = f(0) = 0 \) if \( g(m) = 0 \) (as 0 is a one-element subalgebra of \( M \)). Thus \( f(g(m)) = g(m) \), so \( f \circ g = g \). If \( f(1) = 0 \), then \( f(g(m)) = 0 \) whether \( g(m) = 1 \) or \( g(m) = 0 \). In each case, \( f \circ g \in F \cup \{f_0\} \). \( \square \)

Hence if \( f_0 \) and \( \text{id} \) are included in the fundamental term operations of \( M \), then the unary clone of \( M \) contains exactly the fundamental term operations and the identity. Otherwise, the only things
in the unary clone that are not fundamental term operations are $f_0$ or id or both.

The next lemma provides a simple way of verifying certain subsets of powers of $\{0, 1\}$-valued unary algebras with zero are subuniverses:

**Lemma 2.** If $M$ is a $\{0, 1\}$-valued unary algebra with zero, and $A \subseteq M^I$ such that $A \cap \{0, 1\}^I = \emptyset$, then $M^I \setminus A$ is a subuniverse of $M^I$.

**Proof.** First note that $A$ does not contain the zero element (that is, the element of $M^I$ that is 0 on every coordinate), so $M^I \setminus A$ is non-empty. Pick some $m$ in $M^I \setminus A$, and let $f$ be a non-identity unary term operation of $M^I$. Then $f(m) \in \{0, 1\}^I$. Since $A \cap \{0, 1\}^I = \emptyset$, we have $f(m) \notin A$, so $f(m) \in M^I \setminus A$. Thus $M^I \setminus A$ is a subuniverse of $M^I$. \qed

### 2.4 Topological Structures

Although topology plays a pivotal role in natural duality theory, very little topology is used explicitly in this work. Many of the following definitions are included for the sake of completeness, although the proofs presented in this work can be understood with little-to-no familiarity with topology.

A **topology** $\tau$ on a set $X$ is a collection of subsets of $X$ that satisfy the following properties:

1. Both the empty set $\emptyset$ and the entire set $X$ are in $\tau$,

2. Any union of sets in $\tau$ is also a set in $\tau$, and

3. Any finite intersection of sets in $\tau$ is also in $\tau$.

The sets in $\tau$ are referred to as the **open sets** of the topology, and **closed sets** of $\tau$ are the ones such that their complement is in $\tau$. In the case that $\tau$ contains every subset of $X$, we call $\tau$ the **discrete topology**.
We now make a brief digression back to algebra in order to define algebraic partial and total operations, and algebraic relations. These concepts are necessary to define a topological structure.

Given an algebra $\mathbf{M}$, an **algebraic total operation** (usually referred to as simply an **algebraic operation**) on $\mathbf{M}$ is a homomorphism $g : \mathbf{M}^n \to \mathbf{M}$ for some finite $n$. An **algebraic partial operation** on $\mathbf{M}$ is a homomorphism $h : \mathbf{A} \to \mathbf{M}$ where $\mathbf{A}$ is a subalgebra of $\mathbf{M}^n$ for some finite $n$. An **algebraic relation** on $\mathbf{M}$ is a subuniverse $r$ of $\mathbf{M}^n$ for some finite non-zero $n$. In each case, the value of $n$ is called the **arity** of the operation, partial operation, or relation.

The idea behind algebraic relations, operations, and partial operations is best illustrated by example. Consider the unary algebra $\mathbf{M}_{tot}$ in Figure 2.6. We demonstrate that $A_{tot} = \{(a, b) \in \{0, 1, 2\}^2 \mid a \leq b\}$ (using $\leq$ as the familiar $0 \leq 1 \leq 2$) is an algebraic relation on $\mathbf{M}_{tot}$. Note that any element of $A_{tot}$ is a pair, so the arity of this relation is 2. Thus we wish to show that $A_{tot}$ is a subuniverse of $\mathbf{M}_{tot}^2$. By direct computation, we have

$$\{f_1(x, y) \mid (x, y) \in A_{tot}\} = \{f_2(x, y) \mid (x, y) \in A_{tot}\} = \{(0, 0), (0, 1), (1, 1)\} \subseteq A_{tot},$$

so $A_{tot}$ is closed under both $f_1$ and $f_2$. Hence $A_{tot}$ is a subuniverse of $\mathbf{M}_{tot}^2$. Thus $A_{tot}$ is an algebraic relation on $\mathbf{M}_{tot}$.

For an example of algebraic operations and algebraic partial operations, consider the algebra $\mathbf{M}_{semi}$ in Figure 2.7. We define a partial order on the elements of $\mathbf{M}_{semi}$ using the Hasse diagram in Figure 2.8. Define $\vee : \mathbf{M}_{semi}^2 \to \mathbf{M}_{semi}$ using this partial order (so that for any $x$ in $\mathbf{M}_{semi}$, $0 \vee x = x$, $3 \vee x = 3$, and $1 \vee 2 = 3$). To show that this $\vee$ is algebraic, we need to show that it is a homomorphism from $\mathbf{M}_{semi}^2$ to $\mathbf{M}_{semi}$ by showing it preserves the fundamental operations of $\mathbf{M}_{semi}$. We begin with an operation table for $\vee$, and apply $f_1$ to each entry:
Figure 2.6: The algebra $M_{\text{tot}}$ has the canonical $\leq$ on $\{0,1,2\}$ as an algebraic relation.

<table>
<thead>
<tr>
<th>$M_{\text{tot}}$</th>
<th>$f_1$</th>
<th>$f_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
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</tbody>
</table>

<table>
<thead>
<tr>
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<th>$f_2$</th>
<th>$f_3$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 2.7: The algebra $M_{\text{semi}}$ has an algebraic total operation $\lor$ and an algebraic partial operation $\land$.

<table>
<thead>
<tr>
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<th>3</th>
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</table>

$\Rightarrow f_1$

<table>
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<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

As every entry in the second table is consistent with $\lor$, it follows that $\lor$ preserves $f_1$. The argument showing that $\lor$ preserves $f_2$ and $f_3$ is similar. Thus $\lor$ is an algebraic total operation.

If we now consider $\land^*: M_{\text{semi}}^2 \rightarrow M_{\text{semi}}$ (so that for any $x$ in $M_{\text{semi}}$, $0 \land^* x = 0$, $3 \land^* x = x$, and $1 \land^* 2 = 0$), we run into a problem with $1 \land^* 2 = 0$. Indeed, consider $f_3(1 \land^* 2) = f_3(0) = 0$, but $f_3(1) \land^* f_3(2) = 1 \land^* 1 = 1$, so $\land^*$ does not preserve $f_3$ and so is not an algebraic total operation. The case where $\{x,y\} = \{1,2\}$ is the only obstacle to $\land^*$ being algebraic. Removing this problematic case by restricting the domain of $\land^*$ results in an algebraic partial operation. Let $A = M_{\text{semi}}^2 \setminus \{(1,2),(2,1)\}$, and define $\land = \land^*|_A$, that is, $\land$ is $\land^*$ with the domain restricted to $A$. To show that $\land$ is algebraic, we need to show that $A$ is a subuniverse of
and that $\land$ is a homomorphism. Note that $(0,0) \in A$ so $A$ is non-empty. Since $M^{2\text{semi}}$ is $\{0,1\}$-valued and $(M^{2\text{semi}} \setminus A) \cap \{0,1\}^2 = \emptyset$, $A$ is closed under the term operations of $M^{2\text{semi}}$. Hence $A$ is a subuniverse of $M^{2\text{semi}}$. To show that $\land$ is a homomorphism, consider the operation table of $\land$, and apply $f_3$ to each entry:

<table>
<thead>
<tr>
<th></th>
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<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

The entries in the second table are consistent with $\land$, so it follows that $\land$ preserves $f_3$. The argument is similar for showing $\land$ preserves both $f_1$ and $f_2$. Hence $\land$ is an algebraic partial operation.

A topological structure $X = \langle X; G^X, H^X, R^X; \tau \rangle$ consists of an underlying set $X$, a (possibly empty) set of finitary total operations $G^X$ on $X$, a (possibly empty) set of finitary partial operations $H^X$ on $X$, a (possibly empty) set of finitary relations $R^X$ on $X$, and the discrete topology $\tau$ on $X$ (recall the discrete topology consists of all subsets of $X$). In particular, when $M = \langle M; G^M, H^M, R^M; \tau \rangle$
consists of the underlying set \( M \) of an algebra \( \mathbf{M} \), and \( G^M, H^M \) and \( R^M \) consist of algebraic operations, algebraic partial operations and algebraic relations (respectively) on \( \mathbf{M} \), we call \( \mathbf{M} \) an alter ego for \( \mathbf{M} \). We frequently drop the superscripts on \( G, H, \) and \( R \) when it is clear from context on what structure the operations and relations are defined. Note that alter egos are not unique, as \( G, H, \) and \( R \) are simply arbitrary subsets of all possible algebraic operations, partial operations, and relations (respectively) on \( \mathbf{M} \). We use \( R_n \) to denote all algebraic relations on \( \mathbf{M} \) of arity at most \( n \), as this is frequently the choice for algebraic relations in an alter ego.

The topological structures considered in this thesis do not contain partial operations. For this reason, we simplify some of the following discussion of topological structures with respect to partial operations by stating “making appropriate restrictions”. It should be pointed out that these restrictions are neither trivial nor obvious, and a complete treatment can be found in Section 1.4 of [6].

We define direct products of topological structures in much the same way as for algebras. We concern ourselves only with the case where all the topological structures are equal. Suppose we are given a topological structure \( \mathbf{X} = (X; G, H, R; \tau) \) and a set \( J \). As in the algebraic case, the underlying set of \( \mathbf{X}^J \) is the set \( X^J \). We lift the operations, partial operations, and relations from \( G, H, \) and \( R \) in \( \mathbf{X} \) pointwise to \( \mathbf{X}^J \): the tuple \( \langle x_1, x_2, \ldots, x_n \rangle \) of elements from \( X^J \) is in the relation \( r^{X^J} \in R \) iff for each coordinate \( i \), the tuple \( \langle x_1(i), x_2(i), \ldots, x_n(i) \rangle \) of elements from \( X \) is in \( r^{X^J} \). We drop the superscript on \( r \) when it is clear from context whether we are working in \( \mathbf{X} \) or \( \mathbf{X}^J \). For a simple example, suppose \( \mathbf{X} = (\{0, 1, 2\}; A_{tot}, \tau) \) and \( J = \{a, b\} \). Let \( x_1 = (1, 0) \) and \( x_2 = (2, 2) \in X^J \). Then \( \langle x_1(a), x_2(a) \rangle = (1, 2) \in A_{tot}^X \) and \( \langle x_1(b), x_2(b) \rangle = (0, 2) \in A_{tot}^X \), so each coordinate of \( \langle x_1, x_2 \rangle \) is in \( A_{tot}^X \). Thus \( \langle x_1, x_2 \rangle \) is in \( A_{tot}^{X^J} \). Total operations lift in a manner identical to how the fundamental operations lifted for algebras: if \( g^{X^J} \) is an \( n \)-ary operation in \( G \), we define \( g^{X^J} : (X^J)^n \rightarrow X^J \) by

\[
g^{X^J}(x_1, x_2, \ldots, x_n)(i) = g^X(x_1(i), x_2(i), \ldots, x_n(i))
\]
Morphisms play a parallel role in the topological context as homomorphisms in algebra. If a morphism $\alpha$ is invertible and its inverse is also a morphism, then $\alpha$ is an isomorphism.

Using direct products, substructures, and isomorphisms, we are ready to construct the second half of the context of natural duality theory. The topological quasivariety of a topological structure $X$ is the collection of all isomorphic images of topologically-closed substructures of non-empty direct products of $X$. We denote the topological quasivariety of $X$ by $\mathbb{CP}^+(X)$.

### 2.5 Natural Duality Theory

In very simple terms, the idea of natural duality theory is to start with a particular algebra $M = \langle M; F \rangle$ and pick an appropriate alter ego $M = \langle M; G, H, R; \tau \rangle$ such that we can transition between the quasivariety $\mathbb{ISP}(M)$ and the topological quasivariety $\mathbb{CP}^+(M)$. For some algebras, it may be the case this transition is impossible regardless of our choice of alter ego. From this point on, we denote our starting algebra as $M$ and our chosen alter ego as $M$. All other structures, both algebraic and topological, are boldfaced.

For any $A$ in $\mathbb{ISP}(M)$, we define $\text{hom}(A, M)$ to be the set of all homomorphisms from $A$ to $M$. Because homomorphisms are functions, $\text{hom}(A, M)$ is a subset of $M^A$. In fact, it can be shown that the structure $D(A) = \langle \text{hom}(A, M); G, H, R; \tau \rangle$ is a topologically-closed substructure of $M^A$. This follows from $G, H,$ and $R$ being algebraic. Hence $D(A)$ is in the topological quasivariety $\mathbb{CP}^+(M)$. Similarly, for any $X$ in $\mathbb{CP}^+(M)$, we define $\text{hom}(X, M)$ to be the set of all morphisms from $X$ to $M$. Just like homomorphisms, morphisms are functions so $\text{hom}(X, M)$ is a subset of $M^X$, and more specifically, it can be shown that $E(X) = \langle \text{hom}(X, M); F \rangle$ is a subalgebra of $M^X$. We refer to $D(A)$ and $E(X)$ as the dual of $A$ and $X$ respectively. The details of showing $D(A)$ is in $\mathbb{CP}^+(M)$ and $E(X)$ is in $\mathbb{ISP}(M)$ are given in the proof of Theorem 5.2 in [6] (p. 31).
At this stage it seems natural to ask what happens if we start with an algebra $A$ in $\text{ISP}(M)$ and we look at the dual of the dual—that is, what does $ED(A)$ look like? We define the **natural evaluation homomorphism** $e_A : A \to ED(A)$ given by $e_A(a)(x) := x(a)$ for every $a$ in $A$ and $x$ in $\text{hom}(A, M)$. The notation in this definition is a little awkward, so let us take a closer look at it: $D(A)$ has $\text{hom}(A, M)$ as an underlying set. Rewriting $D(A)$ as $\Gamma$, we have $ED(A) = E(\Gamma)$, which has elements from $\text{hom}(\Gamma, M)$, which are (particular) functions from $\Gamma$ to $M$. Thus the elements of $ED(A)$ are elements of $M^\Gamma$. Hence we may think of the elements of $ED(A)$ as being $\Gamma$-tuples of elements of $M$. Since $\Gamma = D(A) = \text{hom}(A, M)$, the elements of $ED(A)$ may be thought of as being tuples of elements of $M$ indexed by homomorphisms from $A$ to $M$. Looking back at the notation $e_A(a)$, we obtain an element in $ED(A)$, which is a tuple of elements of $M$ indexed by homomorphisms. Thus $e_A(a)(x)$ gives us the “$x$th” coordinate of $e_A(a)$. Defining $e_A(a)(x) = x(a)$ is the natural choice of evaluating $x$ at $a$—hence the name “natural evaluation homomorphism”.

It is always the case that $e_A$ is an embedding. If $e_A$ is an isomorphism, then we say that $M$ yields a duality on $A$. When $M$ yields a duality on every algebra in $\text{ISP}(M)$, we say that $M$ dualises $M$, and $M$ is said to be **dualisable**.

Looking at the formal definitions, dualisability may appear to be a particularly arduous concept to work with. There is a particular way to think about it, however, that can be much simpler. In Figure 2.9, we list the elements $\{a_j\}$ of an algebra $A$ in $\text{ISP}(M)$ along the leading row. The leading column lists all homomorphisms $\{\varphi_i\}$ in $\text{hom}(A, M)$. The $\{i, j\}$ entry of the table is the value $\varphi_i(a_j)$. As $\text{hom}(A, M)$ is the underlying set of $D(A)$, the first column lists the elements of $D(A)$. Each column within the table can now be thought of as a morphism in $\text{hom}(D(A), M)$. Label these morphisms as $\{\psi_i\}$. The table may or may not include all morphisms, however. If not, let us label these “missing” morphisms as $\{\gamma_i\}$. In Figure 2.10, we add more columns to include the morphisms in $\{\gamma_i\}$. Note that the column for $\psi_i$ in Figure 2.10 has the same values as the $a_i$.
<table>
<thead>
<tr>
<th>( \varphi_1 )</th>
<th>( \varphi_1(a_1) )</th>
<th>( \varphi_1(a_2) )</th>
<th>( \varphi_1(a_3) )</th>
<th>( \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>( \varphi_2(a_1) )</td>
<td>( \varphi_2(a_2) )</td>
<td>( \varphi_2(a_3) )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( \varphi_3 )</td>
<td>( \varphi_3(a_1) )</td>
<td>( \varphi_3(a_2) )</td>
<td>( \varphi_3(a_3) )</td>
<td>( \ldots )</td>
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<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>

Figure 2.9: The table of homomorphisms from \( A \) to \( M \).

<table>
<thead>
<tr>
<th>( \psi_1 )</th>
<th>( \psi_1(\varphi_1) )</th>
<th>( \psi_2(\varphi_1) )</th>
<th>( \psi_3(\varphi_1) )</th>
<th>( \ldots )</th>
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<td>( \gamma_1(\varphi_3) )</td>
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<td>( \vdots )</td>
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<td>( \vdots )</td>
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Figure 2.10: The table of morphisms from \( D(A) \) to \( M \).
<table>
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</tr>
</tbody>
</table>

Figure 2.11: The three-element escalator algebra.

column in Figure 2.9. The table now includes all morphisms in \( \text{hom}(D(A), M) \), which is precisely the underlying set of \( ED(A) \). If any \( \gamma \)'s had to be added to the table, then \( ED(A) \) contains more than just a copy of \( A \), so \( M \) would not yield a duality on \( A \). If there were no \( \gamma \)'s to add to the table, then every element of \( A \) corresponds exactly to an element of \( ED(A) \), and hence \( M \) would yield a duality on \( A \).

Two of the most famous examples of dualisability are M. H. Stone's duality for Boolean algebras, and H. A. Priestley's duality for bounded distributive lattices. Stone's duality can be built from the two-element Boolean algebra \( B = \langle \{0,1\}; \land, \lor, '0, 1 \rangle \) which is dualised by the alter ego \( B = \langle \{0,1\}; \tau \rangle \). Priestley's duality can be built from the two-element lattice \( D = \langle \{0,1\}; \land, \lor, 0, 1 \rangle \) which is dualised by the alter ego \( D = \langle \{0,1\}; \leq, \tau \rangle \).

For an example of dualisability of unary algebras, the three-element escalator algebra shown in Figure 2.11 is dualised by the alter ego \( M = \langle M; \land, \lor, E, R; \tau \rangle \) where \( \land \) and \( \lor \) are the lattice operations on the chain \( 0 < 1 < 2 \), \( E = \{(x,y) \mid (x \leq y) \text{ and } (x,y) \neq (0,2)\} \), and \( R = \{(x,y,z,w) \mid (x \leq y \leq z \leq w) \text{ and } x = y \text{ or } z = w\} \) ([13]).
2.6 Duality Theorems

Trying to demonstrate dualisability directly requires showing that $e_A$ is an isomorphism for every $A$ in $\text{ISP}(M)$. Trying to prove non-dualisability directly only requires finding one algebra in the quasi-variety for which $e_A$ is not an isomorphism, but we need to show that every alter ego does not dualise $M$. In this section, we present several significant theorems in the field of natural duality theory, some of which provide indirect methods for proving either dualisability or non-dualisability.

The first theorem we look at allows us to focus on only the finite algebras in $\text{ISP}(M)$ (It should be pointed out this theorem was also proven independently by R. Willard ([22])):

**Theorem 3 (Duality Compactness Theorem).** ([23]) If $M$ is of finite type and yields a duality on each finite algebra in $\text{ISP}(M)$, then $M$ dualises $M$.

The Independence Theorem states that dualisability does not depend on which algebra is chosen to be the generator of the particular quasi-variety. This theorem, along with the Duality Compactness Theorem are used implicitly throughout this thesis.

**Theorem 4 (Independence Theorem).** ([20],[9]) Let $M$ and $N$ be finite algebras such that $\text{ISP}(M) = \text{ISP}(N)$. If $M$ is dualisable, then $N$ is dualisable.

The main technique used for proving non-dualisability is called the Ghost Element Method found in [8]. In Chapter 3, we use a revision of the Ghost Element Theorem:

**Theorem 5 (Revised Ghost Element Method).** ([5]) Let $M$ be a finite algebra. Let $A$ be a subalgebra of $M^S$ for some set $S$, and let $A_0$ be an infinite subset of $A$. Assume that for each homomorphism $\varphi : A \to M$, the equivalence relation $\ker(\varphi|_{A_0})$ has a unique non-trivial block. For $s \in S$, let $\rho_s := \pi_s|_A : A \to M$ denote the natural projection homomorphism on the $s$-coordinate. Define $\gamma \in M^S$ by $\gamma(s) := \rho_s(a_s)$ where $a_s$ is any element of the non-trivial block of $\ker(\rho_s|_{A_0})$. If $M$ is dualisable, then $\gamma \in A$.  

25
The γ in this theorem is referred to as a "ghost element". This theorem is phrased as a necessary condition for dualisability, and its contrapositive is used as a sufficient condition for non-dualisability: given the hypotheses of the theorem, if a ghost element γ in \( M^5 \) can be found such that γ is not in \( A \), then \( M \) is not dualisable.

The next theorem provides a technique for proving dualisability. If for each \( n \in \mathbb{N} \) and each substructure \( X \) of \( M^n \), every morphism \( \alpha : X \to M \) extends to a term function \( t : M^n \to M \) of the algebra \( M \), then we say that \( M \) satisfies the Interpolation Condition relative to \( M \) (or, alternatively, that the interpolation condition holds). Perhaps a more intuitive way of thinking about the interpolation condition in the context of unary algebras is that for every morphism \( \alpha : X \to M \), there is a term \( t \) of \( M^n \) such that \( \alpha(x) = t(x) \) for every \( x \in X \). The terms of \( M^n \) are term operations of \( M \) composed with some projection map, i.e. if \( t \) is a term of \( M^n \), then there exists some \( f \) in the unary clone of \( M \) and some \( i \leq n \) such that \( t = f \circ \pi_i \). Hence to show that the interpolation condition holds is to show that every morphism \( \alpha : X \to M \) is of the form \( \alpha = f \circ \pi_i \mid_X \). If the only algebraic operations in an alter ego \( M \) are the total ones (that is, \( H \) is empty), then \( M \) is called a total structure. If \( M \) is a total structure with finitely many algebraic relations that satisfies the interpolation condition relative to \( M \), we can make use of the following theorem:

**Theorem 6** (Second Duality Theorem). ([8]) Assume that \( M \) is a total structure with \( R \) finite. If the interpolation condition holds, then \( M \) yields a duality on \( M \).

The full statement of the Second Duality Theorem provides an additional property called injectivity. This additional property is not used in this work, and so is omitted.

A particular application of the Second Duality Theorem is that algebras in which certain binary functions are homomorphisms are necessarily dualisable:
for each \( i \in J \) and \( x_1, x_2, ..., x_n \) in \( X_J \). As was the case with algebras, we drop the superscripts and simply use \( g \) to denote the total operation in both the topological structure and the direct product.

For an example, suppose \( X = \langle \{0, 1, 2, 3\}; \vee; \tau \rangle \) using the \( \vee \) from Figure 2.8, and \( J = \{a, b, c\} \). Taking \( x_1 = (0, 1, 2) \) and \( x_2 = (1, 2, 3) \), we have \( x_1 \vee x_2 = (0, 1, 2) \vee (1, 2, 3) = (0 \vee 1, 1 \vee 2, 2 \vee 3) = (1, 3, 3) \).

Algebraic partial operations lift as algebraic total operations making appropriate restrictions on the domain.

As for the topology for \( X_J \), we take the product topology. The **product topology** on the set \( X_J \) is the collection of all subsets of \( X_J \) of the form \( \prod A_i \) such that each \( A_i \) is open in the topology of \( X \), and \( A_i = X \) for all but finitely many \( i \) in \( J \). Since we are using the discrete topology on \( X \), open sets in \( X_J \) are of the form \( \prod A_i \) where \( A_i = X \) for all but finitely many \( i \) in \( J \) (as every subset of \( X \) is open in the discrete topology).

A **closed substructure** of a topological structure \( X = \langle X; G, H, R; \tau^X \rangle \) is a topological structure \( Y = \langle Y; G, H, R; \tau^Y \rangle \) such that \( Y \) is a topologically closed subset of \( X \) (that is, closed with respect to \( \tau^X \)) that is closed under the operations of \( G \) and (where defined) the partial operations of \( H \).

A map \( f : X \rightarrow Y \) between two topological structures \( X \) and \( Y \) is **continuous** if whenever \( B \) is an open subset of \( Y \), \( f^{-1}(B) = \{a \in X \mid f(a) \in B\} \) is open in \( X \). If \( X = \langle X; G^X, H^X, R^X; \tau \rangle \) and \( Y = \langle Y; G^Y, H^Y, R^Y; \tau \rangle \), we say that \( \alpha : X \rightarrow Y \) is a **morphism** if it is continuous and preserves the total and partial algebraic operations, and the algebraic relations in the following sense:

- for each \( n \)-ary operation \( g \) in \( G \) and \( x_1, ..., x_n \) in \( X \), we have:
  \[
  \alpha(g^X(x_1, ..., x_n)) = g^Y(\alpha(x_1), ..., \alpha(x_n))
  \]

  (preservation of partial algebraic operations is defined similarly with appropriate restrictions on the domain);

- for each \( n \)-ary relation \( r \) in \( R \) and \( (x_1, ..., x_n) \) in \( r^X \), we have \( (\alpha(x_1), ..., \alpha(x_n)) \) in \( r^Y \).
Theorem 7. ([7]) Let $M$ be a finite algebra which has binary homomorphisms $\lor$ and $\land$ such that $(M; \lor, \land)$ is a lattice. Then $M := (M; \lor, \land, R_{2|M|}, \tau)$ yields a duality on $\mathbb{ISP}(M)$.

The Second Duality Theorem is used in Section 4.2. Theorem 7 is also mentioned in Section 4.2.

We now have the necessary background to formulate the results of this thesis.
A Condition for Non-Dualisability

The relation \{(0,0), (0,1), (1,0)\} is used extensively in this work, and appears to be — within our context — strongly tied to non-dualisability. Using the ordering $0 \leq 1$, this relation forms an order ideal of $\{0,1\}^2$ whose Hasse diagram resembles a “v”. Hence we call this relation the v-order ideal relation.

Throughout this chapter, $M$ denotes a \{0,1\}-valued unary algebra with zero. Let $F_1$ denote the set of non-constant unary term operations of $M$. Note that $F_1 = F_0 \cup \{id\}$ where $F_0$ is the set of non-constant, non-identity unary term operations of $M$. Recall that by Lemma 1, the unary clone of $M$ consists of the fundamental operations together with the constant 0-valued function and the identity function. Hence $F_1$ is the unary clone of $M$ without the constant 0-valued function.

3.1 PP-Formulae

A primitive positive formula (or pp-formula) is an existentially quantified conjunction of atomic formulae. For unary algebras, an atomic formula has the form $f(x) \approx g(y)$ for term operations $f$ and $g$ (either or both of which may be the identity operation) and (not necessarily distinct) variables
Given a pp-formula $\Phi(x_{1},...,x_{n})$ and $a_{1},...,a_{n}$ elements in $M$, we say that $\Phi(a_{1},...,a_{n})$ holds in $M$ if substituting each $x_{i}$ in $\Phi$ with the corresponding $a_{i}$ results in a true statement in $M$. The values (possibly only one) of $w$ such that $\Phi(a_{1},...,a_{n})$ holds in $M$ are called witnesses of $\Phi(a_{1},...,a_{n})$. If $b$ is a witness for $\Phi(b_{1},...,b_{n})$ for some elements $b_{1},...,b_{n}$ in $M$, then $b$ is a satisfactory witness of $\Phi(x_{1},...,x_{n})$. We may also say that $b$ satisfies $\Phi$.

Consider the pp-formula $\Phi : \exists w \ [f_{3}(w) \approx f_{2}(w) \& x \approx f_{1}(w) \& y \approx f_{2}(w)]$. Looking at the algebra $M_{1}$ in figure Figure 3.1, the elements 0, 3 and 4 are the only ones satisfying $f_{3}(w) \approx f_{2}(w)$. 

<table>
<thead>
<tr>
<th>$M_{1}$</th>
<th>$f_{1}$</th>
<th>$f_{2}$</th>
<th>$f_{3}$</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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Looking at 0, we have \( f_1(0) = 0 \neq f_2(0) \), so 0 witnesses \( \Phi(0,0) \). Similarly, we have \( f_1(3) = 0 \) and \( f_2(3) = 1 \), so 3 witnesses \( \Phi(0,1) \), and 4 witnesses \( \Phi(1,0) \). Hence 0, 3 and 4 are the satisfactory witnesses of \( \Phi(x,y) \).

An important property of pp-formulae is that we use them to define relations on an algebra. On an algebra \( M \), an n-ary relation \( R \) is **pp-defined by a pp-formula** \( \Phi(x_1, \ldots, x_n) \) when \( (a_1, \ldots, a_n) \in R \) iff \( \Phi(a_1, \ldots, a_n) \) holds in \( M \). Using our above example in \( M \), the pp-formula \( \Phi(x,y) \) pp-defines the relation \( R = \{(0,0), (0,1), (1,0)\} \), as \( \Phi(0,0), \Phi(0,1), \) and \( \Phi(1,0) \) hold in \( M \) and no other \( \Phi(a_1, a_2) \) holds in \( M \).

### 3.2 V-Ghostable

In this section, we use pp-formulae to develop a sufficient condition for non-dualisability.

Let \( M \) be a \( \{0,1\} \)-valued unary algebra with 0. Suppose there exist \( Z \subseteq F_1 \), distinct \( t \) and \( u \in F_1 \setminus Z \), and a possibly empty collection \( \{E_i\} \) of subsets of \( F_1 \) such that we can pp-define the v-order ideal relation \( R = \{(0,0), (0,1), (1,0)\} \) via

\[
\Phi: \exists w \left[ \bigwedge_{z \in Z} (z(w) \approx 0) \& \left[ \bigwedge_{E \in \{E_i\}} \left[ \bigwedge_{d,e \in E} d(w) \approx e(w) \right] \& \left[ x \approx t(w) \right] \& \left[ y \approx u(w) \right] \right] \right]
\]

such that if \( w_1 \) and \( w_2 \) witness the same element of \( R \), then \( w_1 = w_2 \). Then we say \( M \) is a v-ghostable algebra. We also say that \( \Phi \) is a v-ghosting formula for \( M \). This terminology is justified over the next several pages.

**Theorem 8.** V-ghostable algebras are not dualisable.

This theorem is proved using a few lemmas. The idea of this proof is to show that \( M \) satisfies the conditions of Theorem 5 (the Revised Ghost Element Method). We construct a subalgebra \( A \) of \( M^\omega \) and ghost element \( \gamma \) such that \( \gamma \) is not in \( A \).
\begin{align*}
\begin{array}{c|cc}
\text{M} & t & u \\
0 & 0 & 0 \\
1 \neq p & 0 & 1 \\
q & 1 & 0 \\
\end{array}
\end{align*}

Figure 3.2: The behaviour of 0, p, and q under the term operations $t$ and $u$.

For the remainder of this section, assume that $\mathbb{M}$ is a $v$-ghostable algebra, $\Phi$ is is a $v$-ghosting formula for $\mathbb{M}$ as in Formula 3.1, and $R$ is the relation $\{(0,0), (0,1), (1,0)\}$. Since $\mathbb{M}$ has a 0, it must be the unique element witnessing $(0,0) \in R$. Let $p$ be the unique witness of $(0,1)$ and $q$ be the unique witness of $(1,0)$. Thus $t(p) = 0 = u(q)$ and $u(p) = 1 = t(q)$. Note that $p$ and $q$ are defined symmetrically and not both of $p$ and $q$ can be 1, so we may assume without loss of generality that $p \neq 1$. Also note that neither $p$ nor $q$ can be 0. Figure 3.2 summarizes the information in this paragraph.

For $i, j \geq 0$ and $i \neq j$, define $a_{ij} \in \mathbb{M}^\omega$ by

\begin{align*}
a_{ij}(s) &= \begin{cases} 
  p & \text{if } s = i, \\
  q & \text{if } s = j, \\
  0 & \text{otherwise.}
\end{cases}
\end{align*}

Note that for all $i \neq j$, we have

\begin{align*}
t(a_{ij})(s) &= \begin{cases} 
  1 & \text{if } s = j \\
  0 & \text{otherwise}
\end{cases}
\quad \text{and} \quad
u(a_{ij})(s) &= \begin{cases} 
  1 & \text{if } s = i \\
  0 & \text{otherwise.}
\end{cases}
\end{align*}

Let $A_1 = \{a_{ij} \mid i \neq j \text{ and } i, j \geq 1\}$, $A = \text{Sg}_\mathbb{M}(A_1)$, and $A_0 = \{a_{1j} \mid j > 1\} \subseteq A_1$.

\textbf{Lemma 9.} Let $\varphi \in \text{hom}(A, \mathbb{M})$. Then $\varphi(A_1) \subseteq \{0, p, q\}$.
Figure 3.3: A few elements of the form $a_{ij}$. Each column corresponds to an element, each row to a coordinate (the first row being the $0^{th}$ coordinate).

**Proof.** We repeatedly use the fact that $0$, $p$, and $q$ are the witnesses to a $v$-ghosting formula for $M$.

Consider some $a_{ij} \in A_1$. Then for all $z \in Z$, we have $z(a_{ij}) = f_0(a_{ij})$ since $a_{ij} \in \{0,p,q\}^\omega$ and $z(\{0,p,q\}) = \{0\}$ by Formula 3.1. Thus

$$z(\varphi(a_{ij})) = \varphi(z(a_{ij})) = \varphi(f_0(a_{ij})) = f_0(\varphi(a_{ij})) = 0,$$

so $\varphi(a_{ij})$ satisfies $\bigwedge_{z \in Z} z(w) \approx 0$. Consider some $E \in \{E_i\}$ and $d,e \in E$. Since $a_{ij} \in \{0,p,q\}^\omega$ and $d(w) = e(w)$ for $w \in \{0,p,q\}$ by Formula 3.1, we have $d(a_{ij}) = e(a_{ij})$. Then

$$d(\varphi(a_{ij})) = \varphi(d(a_{ij})) = \varphi(e(a_{ij})) = e(\varphi(a_{ij}))$$

so $\varphi(a_{ij})$ satisfies $\bigwedge_{E \in \{E_i\}} \bigwedge_{d,e \in E} d(w) \approx e(w)]$. Thus $\varphi(a_{ij})$ satisfies $\Phi$, so $\varphi(a_{ij})$ witnesses some element of $R$. But since each element of $R$ is uniquely witnessed by one of $0$, $p$, or $q$, we have $\varphi(a_{ij}) \in \{0,p,q\}$. Thus $\varphi(A_1) \subseteq \{0,p,q\}$, as desired. □

<table>
<thead>
<tr>
<th>$a_{01}$</th>
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<th>$a_{42}$</th>
<th>$a_{54}$</th>
<th>$\ldots$</th>
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<tbody>
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| $\vdots$| $\vdots$| $\vdots$| $\vdots$| $\vdots$|
Lemma 10. If $\varphi \in \text{hom}(A, M)$, then $\varphi|_{A_0}$ is the restriction a projection map.

Proof. First note that if $\varphi(A_0) = \{0\}$, then $\varphi|_{A_0} = \pi_0|_{A_0}$. We handle the remaining cases separately.

Using notation similar to that found in [5], we define $0^1_n$ for $n \geq 1$ by

$$0^1_n(s) = \begin{cases} 1 & \text{if } s = n \\ 0 & \text{otherwise,} \end{cases}$$

so that $t(a_{1j}) = 0^1_j$ and $u(a_{1j}) = 0^1_j$. In particular, for $a_{1j}$ in $A_0$, we have $u(a_{1j}) = 0^1_j$. Note that since $a_{1n}$ is in $A_0 \subseteq A$ and $t(a_{1n}) = 0^1_n$, we have that $0^1_n$ is in $A$. Figures 3.3 and 3.4 help illustrate these elements. Throughout we use the fact that $\varphi(A_1) \subseteq \{0, p, q\}$.

Case 1: $q \in \varphi(A_0)$

As $q \in \varphi(A_0)$, there exists $j$ such that $\varphi(a_{1j}) = q$. We show $\varphi|_{A_0} = \pi_j|_{A_0}$ by considering the behaviour of elements $a_{1k}$ and $a_{jk}$ with $k \neq j$:

$$u(\varphi(a_{1k})) = \varphi(u(a_{1k})) = \varphi(0^1_j) = \varphi(u(a_{1j})) = u(\varphi(a_{1j})) = u(q) = 0,$$

so $\varphi(a_{1k}) \in \{0, q\}$ (since $u(p) = 1$). Now consider the element $a_{jk}$:

$$u(\varphi(a_{jk})) = \varphi(u(a_{jk})) = \varphi(0^1_j) = \varphi(t(a_{1j})) = t(\varphi(a_{1j})) = t(q) = 1.$$

Since $p$ is the only element of $\{0, p, q\}$ satisfying $u(x) \approx 1$, we have $\varphi(a_{jk}) = p$. Finally,

$$t(\varphi(a_{1k})) = \varphi(t(a_{1k})) = \varphi(0^1_k) = \varphi(t(a_{jk})) = t(\varphi(a_{jk})) = t(p) = 0.$$
so \( \varphi(a_{1k}) \in \{0, p\} \). Thus \( \varphi(a_{1k}) \in \{0, q\} \cap \{0, p\} \), so \( \varphi(a_{1k}) = 0 \). Hence \( \varphi|_{A_0} = \pi_j|_{A_0} \).

**Case 2:** \( p \in \varphi(A_0) \)

Pick \( a_{1j} \in A_0 \) such that \( \varphi(a_{1j}) = p \). Then for any \( a_{1k} \in A_0 \) we have

\[
\varphi(\varphi(a_{1k})) = \varphi(\varphi(a_{1k})) = \varphi(0^j) = \varphi(u(a_{1j})) = \varphi(\varphi(a_{1j})) = u(p) = 1,
\]

which implies that \( \varphi(a_{1k}) = p \). Hence \( \varphi(A_0) = \{p\} \), so \( \varphi|_{A_0} = \pi_1|_{A_0} \).

This exhausts the possible cases, proving the lemma.

We are now ready to construct the ghost element of \( A \), which we use to prove Theorem 8.

**Proof.** (of Theorem 8) By the above lemma, the homomorphisms \( \varphi \) from \( A \) to \( M \) are the projection maps when restricted to \( A_0 \). If \( \varphi|_{A_0} = \pi_0|_{A_0} \) or \( \varphi|_{A_0} = \pi_1|_{A_0} \), then \( \varphi(a) = \varphi(b) \) for all \( a, b \in A_0 \), so \( \ker(\varphi|_{A_0}) \) has a unique non-trivial block which is all of \( A_0 \). Note that on this non-trivial block \( (A_0) \), we have that \( \pi_0 \) takes the value 0, and \( \pi_1 \) takes the value \( p \). If \( \varphi|_{A_0} = \pi_j|_{A_0} \) for some \( j > 1 \), then \( \varphi(a_{1j}) = q \) and \( \varphi(a_{1k}) = 0 \) for all \( k \neq j \), so \( \ker(\varphi|_{A_0}) \) has a unique non-trivial block which is \( A_0 \setminus \{a_{1j}\} \). For each \( j > 1 \), we have that \( \pi_j \) takes the value 0 on \( A_0 \setminus \{a_{1j}\} \). See Figure 3.5. Thus the ghost element is

\[
\gamma(s) = \begin{cases} 
  p & \text{if } s = 1 \\
  0 & \text{otherwise.}
\end{cases}
\]

For any generating element \( a_{ij} \in A \), we have \( a_{ij}(j) = q \). No such coordinate exists for \( \gamma \), so \( \gamma \) is not a generating element of \( A \). Furthermore, since \( p \notin \{0, 1\} \) and \( M \) is \( \{0, 1\} \)-valued, there is no term \( f \) in the clone of \( M \) such that \( f(a) = \gamma \). Thus \( \gamma \notin A \). Hence, by the contrapositive of Theorem 5 (the Revised Ghost Element Method), \( M \) is not dualisable, proving Theorem 8.

The proof of Theorem 8 may not appear particularly intuitive at first glance, but it can be thought of in terms of “eliminating” elements from \( M \) until only 0, \( p \), and \( q \) remain. Each piece of
the \( v \)-ghosting formula gives a different method of elimination. In the following section, Theorem 8 is applied to categorise the dualisability of a class of unary algebras. In section 5.1, Theorem 8 is translated into a simple test for non-dualisability in certain circumstances, which demonstrates the usefulness of this theorem.
Chapter 4

Order Ideals

Recall in Section 2.3 we discussed that for a \( \{0,1\} \)-valued unary algebra with zero \( M \), the rows of \( M \) form a subset of \( \{0,1\}^n \) for some \( n \). For the remainder of this work, whenever we say that the rows of an algebra form an order ideal, we mean that it forms an order ideal of \( \{0,1\}^n \) using the pointwise ordering \( 0 < 1 \). In this chapter, we characterise the dualisability of \( \{0,1\} \)-valued unary algebras with zero whose rows are uniquely witnessed when the rows form an order ideal.

Recall that given a partial order, an order ideal is closed downwards (and hence is closed under \( \land \)), and a lattice is closed under both \( \land \) and \( \lor \). Throughout this chapter, we assume \( M \) has uniquely witnessed rows.

**Theorem 11.** Let \( M \) be a \( \{0,1\} \)-valued unary algebra with zero such that \( \text{Rows}(M) \) is uniquely witnessed and forms an order ideal under \( 0 \leq 1 \). Then \( M \) is dualisable if and only if \( \text{Rows}(M) \) is a lattice order.

The proof of this theorem is developed in Sections 4.1 and 4.2. In Section 4.1, we apply Theorem 8 to prove non-dualisability when \( \text{Rows}(M) \) is uniquely witnessed and does not form a lattice order. In Section 4.2 we apply Theorem 6 to prove dualisability when \( \text{Rows}(M) \) is uniquely
witnessed and does form a lattice order.

4.1 Non-Lattice Order Ideals

Throughout this section, we assume that $M$ is a $\{0,1\}$-valued unary algebra with 0. To prove Theorem 11, we first consider when $\text{Rows}(M)$ is not a lattice order. In this case, we show that $M$ is a $v$-ghostable algebra.

In this section, we use an alternate indexing of $\text{Rows}(M)$. Note that each element of $\text{Rows}(M)$ corresponds to a particular element of $M$. By appropriately ordering $F_0$ (recall that $F_0$ is the non-constant, non-identity term operations), we may consider a tuple $\mu$ of $\text{Rows}(M)$ as $\mu = \langle f(m) \rangle_{f \in F_0}$ where $\mu = \text{row}(m)$. Hence the tuples of $\text{Rows}(M)$ are indexed by $F_0$, and we can consider $\mu(f)$ instead of $f(m)$. This eases notation, as we need not specify the corresponding element of $M$ when discussing the coordinates of elements of $\text{Rows}(M)$.

Considering the partial order on $\text{Rows}(M)$, if $\text{Rows}(M)$ forms an order ideal and there is a maximum element, then $\text{Rows}(M)$ is a lattice order. Indeed, suppose that $v$ is the maximum element, and let $\wedge_L$ and $\vee_L$ be the meet and join realised in the lattice order $L = \langle \{0,1\}^{F_0}, \leq \rangle$ respectively. For any $a, b \in \text{Rows}(M)$, we have $a \leq v$ and $b \leq v$, so

$$a \vee_L b \leq v \vee_L v = v \in \text{Rows}(M).$$

Since $\text{Rows}(M)$ is an order ideal, the above line shows $a \vee_L b$ is in $\text{Rows}(M)$. Thus $\text{Rows}(M)$ is closed under $\vee_L$. As $\text{Rows}(M)$ is an order ideal, it must also be closed under $\wedge_L$ (since $a \wedge_L a \leq a$). Hence $\text{Rows}(M)$ is a lattice order if and only if it has a maximum element. We may therefore assume that if $\text{Rows}(M)$ is an order ideal that is not a lattice order, then it has at least two distinct maximal elements.
Let $\tau$ and $\nu$ be two distinct maximal elements in $\text{Rows}(M)$. For $(i, j)$ in $\{0, 1\}^2$, let $E_{ij} = \{ f \in F_0 \mid \tau(f) = i \text{ and } \nu(f) = j \}$, and let $E_{*1} = E_{01} \cup E_{11} = \{ f \in F_0 \mid \nu(f) = 1 \}$. See Figure 4.1 on the following page.

**Lemma 12.** $E_{*1}$ and $E_{10}$ are non-empty.

*Proof.* First consider $E_{*1} = \{ f \in F_0 \mid \nu(f) = 1 \}$. If there is no $f \in F_0$ such that $\nu(f) = 1$, then $\nu = \text{row}(0)$. Thus $\nu \leq \tau$, contradicting the maximality of $\nu$. Thus there is an $f \in F_0$ with $\nu(f) = 1$, so $E_{*1}$ is non-empty.

To show $E_{10}$ is non-empty, note that since $\tau$ and $\nu$ are distinct maximal elements, we have $\tau \not\leq \nu$. Thus there is a $g$ such that $\tau(g) \not\leq \nu(g)$. This occurs only when $\tau(g) = 1$ and $\nu(g) = 0$. Hence $g$ is in $E_{10}$, so $E_{10}$ is non-empty. $\square$

Define $\sigma$ by

$$
\sigma(s) = \begin{cases} 
0 & \text{if } s \in E_{11} \\
\tau(s) & \text{otherwise.}
\end{cases}
$$

Clearly $\sigma \in \text{Rows}(M)$ since $\sigma(s) \leq \tau(s)$ for every $s$. Pick $p_\nu, p_\sigma \in M$ such that $\text{row}(p_\nu) = \nu$ and $\text{row}(p_\sigma) = \sigma$. Fix $e_{10}$ and $e_{*1}$ operations in $E_{10}$ and $E_{*1}$ respectively. Notice that for any $f \in F_0$, we have $f(p_\nu) = \nu(f)$, and similarly $f(p_\sigma) = \sigma(f)$ (this sort of back-and-forth is used extensively in the following lemma). Consider the pp-formula

$$
\Phi : \exists w \left[ \forall z \in E_{00} \left[ \lnot \exists w \left[ \forall t \in E_{10} \left[ t(w) \approx t'(w) \right] \& \forall u, u' \in E_{*1} \left[ u(w) \equiv u'(w) \right] \right] \right] \right.
$$

$$
\& \left[ \forall t \in E_{10} \left[ t(w) \approx t'(w) \right] \& \forall u, u' \in E_{*1} \left[ u(w) \equiv u'(w) \right] \right] \left. \right] .
$$

The next lemma shows that this is a $\nu$-ghosting formula for $M$ using $Z = E_{00}, E = \{E_{10}, E_{*1}\}, t = e_{10}$ and $u = e_{*1}$.

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Lemma 13. The elements 0, $p_v$, and $p_\sigma$ witness (respectively) $\Phi(0,0)$, $\Phi(0,1)$, and $\Phi(1,0)$ in Formula 4.1. Furthermore, 0, $p_v$, and $p_\sigma$ are the only elements of $M$ that are satisfactory witnesses of $\Phi(x,y)$.

Proof. As $f(0) = 0$ for every term operation in $M$, it is clear that 0 is a satisfactory witness of $\Phi(x,y)$ and that 0 witnesses $\Phi(0,0)$. In addition, since $v(z) = \tau(z) = 0$ for every $z$ in $E_{00}$, we have $\sigma(z) = 0$ and $v(z) = 0$. Thus both $p_v$ and $p_\sigma$ satisfy $\wedge_{z \in E_{00}} z(w) \approx 0$ as

$$f(p_v) = 0 \text{ iff } v(f) = 0$$

and $f(p_\sigma) = 0 \text{ iff } \sigma(f) = 0$.

From the definition of $E_{10}$ we have $v(t) = 0$ for all $t \in E_{10}$, so $p_v$ satisfies $\wedge_{t, t' \in E_{10}} t(w) \approx t'(w)$. Note that for any $t$ in $E_{10}$ we have both $t \notin E_{00}$ (since $E_{10}$ and $E_{00}$ are disjoint) and $t \notin E_{11}$ (since $v(t) \neq 1$). Thus $v(t) \neq \tau(t) = \sigma(t)$. Thus $\sigma(t) \neq 0$; in other words, $\sigma(t) = 1$. As this is true for every $t$ in $E_{10}$, $p_\sigma$ satisfies $\wedge_{t, t' \in T} t(w) \approx t'(w)$.

A nearly identical argument works for $E_{11}$. From the definition of $E_{11}$, we have $v(u) = 1$ for every $u \in E_{11}$, so $p_v$ satisfies $\wedge_{u, u' \in U} u(w) \approx u'(w)$. For every $u$ in $E_{11}$, we have either $\tau(u) = 1$ or $\tau(u) = 0$. If $\tau(u) = 1$, then $u \in E_{11}$, so $\sigma(u) = 0$. If $\tau(u) = 0$, then $\sigma(u) = 0$ as $\sigma(u) \leq \tau(u)$. Thus $p_\sigma$ satisfies $\wedge_{u, u' \in E_{11}} u(w) \approx u'(w)$. Therefore 0, $p_v$, and $p_\sigma$ satisfy Formula 4.1.

Figure 4.1: The values of $\tau$, $v$, and $\sigma$ on the sets $E_{00}$, $E_{10}$, $E_{01}$ and $E_{11}$.

<table>
<thead>
<tr>
<th></th>
<th>$E_{00}$</th>
<th>$E_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$v$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>
Recall that $e_{10}$ and $e_{*1}$ were arbitrary but particular elements of $E_{10}$ and $E_{*1}$ respectively. From the discussion above, we have in particular that $\nu(e_{10}) = 0$ and $\nu(e_{*1}) = 1$, so $e_{10}(p_\nu) = 0$ and $e_{*1}(p_\nu) = 1$. Therefore $p_\nu$ witnesses $\Phi(0,1)$. Similarly, $e_{10}(p_\sigma) = 1$ and $e_{*1}(p_\sigma) = 0$, so $p_\sigma$ witnesses $\Phi(1,0)$. Also, $e_{10}(0) = 0 = e_{*1}(0)$, so 0 witnesses $\Phi(0,0)$. The last step to show that $M$ satisfies Theorem 8 is that 0, $p_\nu$, and $p_\sigma$ are the only elements satisfying Formula 4.1.

We initially assume that $E_{00}$ is non-empty. The trick here is to note that $E_{10} \cup E_{*1} = E_{10} \cup E_{01} \cup E_{11} = F_0 \setminus E_{00}$ so that $E_{00} \cup E_{10} \cup E_{*1} = E_0 \cup (F_0 \setminus E_0) = F_0$. More specifically, this means that the row of any satisfactory witness $m$ in $M$ can be completely described by the tuple $(z(m), t(m), u(m))$ where $z$, $t$, and $u$ are any elements from $E_{00}$, $E_{10}$ and $E_{*1}$ respectively. Suppose $m \in M$ satisfies Formula 4.1. Then for every $z \in E_{00}$, every $t \in E_{10}$, and every $u \in E_{*1}$ we have either $(z(m), t(m), u(m)) = (0, 0, 0); (0, 1, 0); (0, 0, 1);$ or $(0, 1, 1)$. In the first three cases, because the rows of $M$ are uniquely witnessed, $m = 0$, $p_\sigma$, or $p_\nu$ respectively. If $(z(m), t(m), u(m)) = (0, 1, 1)$, then $row(m) > \nu$ (using our observation that these three-tuples completely characterise the rows), contradicting the maximality of $\nu$. Thus 0, $p_\sigma$, and $p_\nu$ are the only elements satisfying Formula 4.1. If $E_{00}$ is empty, the same argument applies by omitting $z$.

Thus $M$ satisfies the conditions of Theorem 8, so $M$ is not dualisable if its rows form an order ideal that is not a lattice order.

### 4.2 Lattice Orders

In this section, we prove that $\{0, 1\}$-valued unary algebras with 0 with unique witnesses whose rows form a lattice order are dualisable.
Recall that Theorem 7 states that a finite algebra that has binary homomorphisms that form a lattice structure is dualisable. Although the result from this section follows almost immediately from Theorem 7, we provide a detailed proof because Theorem 11 is the only positive result for dualisability in this thesis. For this section, we assume $\text{Rows}(M)$ forms a lattice order under the point-wise ordering with $0 < 1$. We use $\wedge$ and $\vee$ to denote the lattice operations either on $\{0,1\}$ or $\text{Rows}(M)$ under this ordering. We need not worry about the ambiguity of the domain of $\wedge$ and $\vee$, as it is clear from context.

Recall the Interpolation Condition: if for each $n \in \mathbb{N}$ and each substructure $X$ of $M^n$, every morphism $\alpha : X \rightarrow M$ extends to a term function $t : M^n \rightarrow M$ of the algebra $M$, then $M$ satisfies the interpolation condition relative to $M$.

Perhaps a more intuitive way of thinking about the Interpolation Condition in the context of unary $\{0,1\}$-valued unary algebras with zero is that for every morphism $\alpha : X \rightarrow M$, there is a term $t$ of $M^n$ such that $\alpha(x) = t(x)$ for every $x \in X$. The terms of $M^n$ are terms of $M$ composed with some projection map, i.e. if $t$ is a term of $M^n$, then there exists some $f$ in the unary clone of $M$ and some $i < n$ such that $t = f \circ \pi_i$ where $\pi_i : M^n \rightarrow M$ is the $i^{th}$ projection map. Hence to show that the Interpolation Condition holds is to show that every morphism $\alpha : X \rightarrow M$ is of the form $\alpha = f \circ \pi_i |_X$.

Once we establish that the Interpolation Condition holds, we make use of the Second Duality Theorem:

**Theorem 6.** ([8]) Assume that $M$ is a total structure with finitely many algebraic operations. If the interpolation condition holds, then $M$ yields a duality on $M$.

To apply this theorem, we must first construct our alter ego $M$. This construction is fairly similar to the one found in Theorem 2.1 in [13]. In this construction, $M$ contains both relations and total operations. We consider three different cases, dependent on the size of the range of the
morphism \(\alpha\). In each case, we make use of a particular kind of algebraic relation, and so we construct three categories of relations. Before we can construct these relations, we must establish lattice operations on \(M\) and demonstrate that they are algebraic operations.

Since we already have a lattice structure on \(\text{Rows}(M)\), we construct a lattice structure on \(M\) in terms of \(\text{Rows}(M)\) (in a fairly canonical manner). Define \(\wedge^*\) on \(M\) by
\[
a \wedge^* b = c \text{ iff } \text{row}(a) \wedge \text{row}(b) = \text{row}(c).
\]
The operation \(\wedge^*\) is well defined since \(M\) has unique rows. Note that because \(\text{Rows}(M)\) is a lattice order, for all \(a, b \in M\), we have \(\text{row}(a) \wedge \text{row}(b) \in \text{Rows}(M)\) and hence there exists \(c \in M\) such that \(\text{row}(c) = \text{row}(a) \wedge \text{row}(b)\). Thus \(M\) is closed under \(\wedge^*\), so \(\wedge^*\) is a total operation on \(M\). We define \(\lor^*\) on \(M\) mutatis mutandis, and so it is also a total operation on \(M\).

**Lemma 14.** Both \(\wedge^*\) and \(\lor^*\) are algebraic operations of \(M\).

**Proof.** We prove that \(\wedge^* : M^2 \to M\) is a homomorphism. The proof for \(\lor^*\) is similar.

Let \(a, b \in M\), \(a \wedge^* b = c\), and \(f \in \text{clone}_u(M)\). If \(f\) is constant, then it must be the constant 0 as \(\{0\}\) is a subuniverse of \(M\). Thus \(f(a) \wedge^* f(b) = 0 \wedge 0 = 0 = f(a \wedge^* b)\). If \(f\) is the identity term, then \(f(a) \wedge^* f(b) = a \wedge^* b = c = f(c) = f(a \wedge^* b)\). Finally, if \(f\) is a non-identity, non-constant term, then we may consider \(f\) as a coordinate of \(\text{Rows}(M)\). Thus
\[
f(a) \wedge^* f(b) = (\text{row}(a))(f) \wedge (\text{row}(b))(f) = (\text{row}(a) \wedge \text{row}(b))(f)
\]
\[
= (\text{row}(c))(f) = f(c) = f(a \wedge^* b).
\]
Hence \(\wedge^*\) preserves the term operations of \(M\), and \(\wedge^*\) is an algebraic operation. \(\square\)

As we have constructed a lattice structure on \(M\), let \(\leq\) denote the associated partial order. To be certain that there is no ambiguity, we should really use \(\leq^M\) to denote this partial order. However, it will be clear from context whether \(\leq\) is being used on \(\{0, 1\}\) or \(M\). We are ready to construct the necessary relations on \(M\).
Let \( R_1 = \{ (a, b) \in M^2 \mid a \leq b \text{ and } \forall f \in \text{clone}_u(M), f((a, b)) \neq (1, 1) \} \). Let \( R_2 = \{ (a, b, c, d) \in M^4 \mid a \leq b \leq c \leq d \text{ and } \forall f \in \text{clone}_u(M), f((a, b, c, d)) \neq (0, 0, 1, 1) \} \). For the last collection of relations, introduce an arbitrary linear order \( \leq \) on \( M \). Let \( \mathcal{S} = \{ S \subseteq M \mid S \setminus \{0, 1\} \neq \emptyset \} \). Fix \( S \) in \( \mathcal{S} \) and enumerate the elements of \( S \) as \( m_1, m_2, \ldots, m_k \) with \( m_1 \leq m_2 \leq \ldots \leq m_k \). Let \( t_S = (m_1, m_1, \ldots, m_k, m_k) \in M^{2k} \) and \( R_S = M^{2k} \setminus \{t_S\} \).

**Lemma 15.** \( \mathcal{R} = \bigcup_{S \in \mathcal{S}} \{R_S\} \cup \{R_1, R_2\} \) is a set of algebraic relations.

**Proof.** First consider \( R_S \) for some \( S \) in \( \mathcal{S} \). Pick \( c \) in \( S \setminus \{0, 1\} \). Then there exists some coordinate \( \mu \) for which \( \pi_\mu(t_S) = c \). Suppose \( f \) is any non-identity term in the unary clone of \( M \). Then because \( M \) is \( \{0, 1\} \)-valued, given any \( m \in M^{2k} \), we have \( f(m) \in \{0, 1\}^{2k} \), and \( \pi_\mu(f(m)) \in \{0, 1\} \). But \( \pi_\mu(t_S) = c \notin \{0, 1\} \), so \( f(m) \neq t_S \) for any non-identity term \( f \) and element \( m \). Therefore \( R_S = M^{2k} \setminus \{t_S\} \) is closed under the operations of \( M \), and therefore is algebraic for any \( S \) in \( \mathcal{S} \).

Now consider \( R_2 \). Let \( r = (r_1, r_2, r_3, r_4) \in R_2 \). For \( i \leq j \), we have \( r_i \leq r_j \), so \( r_i \wedge r_j = r_i \). Suppose \( g \) is in the unary clone of \( M \). Then \( g(r_i) = g(r_j \wedge r_i) = g(r_j) \wedge^* g(r_i) \) since \( \wedge^* \) is an algebraic operation. Therefore \( g(r_i) \leq g(r_j) \). Furthermore, for any \( f \in \text{clone}_u(M) \), \( f \circ g \) is in \( \text{clone}_u(M) \), so \( f(g(r)) \neq (0, 0, 1, 1) \). Therefore \( g(r) \in R_2 \), so \( R_2 \) is a subalgebra of \( M^4 \) as desired.

The proof for \( R_1 \) is nearly identical as for \( R_2 \). Let \( r = (r_1, r_2) \in R_1 \), so \( r_1 \leq r_2 \) and \( r_1 \wedge r_2 = r_1 \). Suppose \( g \) is some term in the unary clone of \( M \). Then because \( \wedge^* \) is algebraic in \( M \), \( g(r_1) = g(r_1 \wedge^* r_2) = g(r_1) \wedge^* g(r_2) \). Hence \( g(r_1) \leq g(r_2) \). Also, for any \( f \in \text{clone}_u(M) \), \( f \circ g \) is in the unary clone of \( M \), so \( f(g(r)) \neq (1, 1) \). Hence \( g(r) \in R_1 \), so \( R_1 \) is algebraic. \( \square \)

This next lemma is required as the interpolation condition requires that the collection of algebraic relations used in the alter ego is finite.

**Lemma 16.** \( \mathcal{R} \) is finite.
Figure 4.2: The relationships between $b_0$, $a_0$, $b_1$, $a_0^1$ and $a_1$. The ellipses (from left to right) represent the elements sent by $\alpha$ to 0 and 1.

Proof. Since $\{R_1, R_2\}$ contains only two elements, we need only show that $\bigcup S\{R_S\}$ is finite. Note that each $R_S$ depends only on $S$. Since $S \subseteq M$, we have that

$$|\{S \subseteq M : S \not\subseteq \{0,1\}\}| \leq 2^{|M|}.$$  Since $M$ is finite, $2^{|M|}$ is finite, and so $\bigcup S\{R_S\}$ is finite

Let $M = \langle M; \{\land^*, \lor^*\}, \mathcal{R}, \tau \rangle$. We need to show that $M$ satisfies the conditions of the Second Duality Theorem. Since $M$ contains no partial operations, it is a total structure.

Lemma 17. $M$ satisfies the interpolation condition relative to $M$.

Proof. Pick $X$ a substructure of $M^n$ for some finite $n$ and a morphism $\alpha : X \rightarrow M$. Then $X$ is necessarily finite. Also, since we have defined a lattice structure on $M$, $X$ also has a lattice structure and we may perform all the familiar lattice operations such as finding the supremum and infimum of sets of elements. We use $R_i^M$ and $R_i^X$ to denote $R_i$ realised in $M$ and $X$ respectively. We break this proof up into cases based on the range of $\alpha$. For each $i$ in $M$ with $\{x \in X \mid \alpha(x) = i\}$ non-empty, let $a_i = \sup\{x \in X \mid \alpha(x) = i\}$ and $b_i = \inf\{x \in X \mid \alpha(x) = i\}$. Figure 4.2 helps illustrate the relationships between these elements.
Case 1: \( \alpha(X) = \{0\} \)

Since \( f_0 \) is the constant 0 function on \( M \), \( \alpha = f_0 \circ \pi_1 \mid_X \).

Case 2: \( \alpha(X) = \{0, 1\} \)

Let \( a_0^1 = a_0 \lor b_1 \) and \( s = \langle b_0, a_0, a_0^1, a_1 \rangle \). Note that \( b_0 \leq a_0 \leq a_0^1 \leq a_1 \). As \( a_0^1 \geq b_1 \), \( \alpha(a_0^1) = 1 \), it follows that \( \alpha(s) = \langle 0, 0, 1, 1 \rangle \). By definition of \( R^M_2 \), we have \( \alpha(s) = \langle 0, 0, 1, 1 \rangle \) which is not in \( R^M_2 \), and since morphisms preserve \( R \), \( s \not\in R^X_2 \). Therefore there must be some coordinate, say \( \ell \), for which \( s(\ell) \) fails to be in \( R^M_2 \). In other words, \( s(\ell) = \langle b_0(\ell), a_0(\ell), a_0^1(\ell), a_1(\ell) \rangle \not\in R^M_2 \).

Since \( s(\ell) \) has the form of an element from \( R^M_2 \) (i.e. \( s(\ell) \) is a 4-tuple in ascending order), but is not in \( R^M_2 \), it must be the case that there exists \( f \in \text{clone}_u(M) \) such that \( f(s(\ell)) = \langle 0, 0, 1, 1 \rangle = \alpha(s) \). Since \( a_0^1(\ell) = a_0(\ell) \lor b_1(\ell) \), we have \( f(a_0^1(\ell)) = f(a_0(\ell)) \lor f(b_1(\ell)) \) as \( \lor \) is a homomorphism. That is \( 1 = 0 \lor f(b_1(\ell)) \), so \( f(b_1(\ell)) = 1 \). Since for every \( x_1 \in \alpha^{-1}(\{1\}) \) we have that \( b_1 \leq x_1 \leq a_1 \), we must also have \( f(b_1) \leq f(x_1) \leq f(a_1) \). This series of inequalities must hold on each coordinate—specifically it must hold on \( \ell \), so we have \( 1 = f(b_1(\ell)) \leq f(x_1(\ell)) \leq f(a_1(\ell)) = 1 \). Thus \( f(x_1(\ell)) = 1 \) for every \( x_1 \in \alpha^{-1}(\{1\}) \).

For \( x_0 \in \alpha^{-1}(\{0\}) \), we have directly from \( f(s(\ell)) = \langle 0, 0, 1, 1 \rangle \) that \( f(b_0(\ell)) = 0 = f(a_0(\ell)) \). Thus \( 0 = f(b_0(\ell)) \leq f(x_0(\ell)) \leq f(a_0(\ell)) = 0 \), so \( f(x_0(\ell)) = 0 \). Hence for every \( x \) in \( X \), we have \( \alpha(x) = f(x(\ell)) \), so \( \alpha = f \circ \pi_\ell \mid_X \) where \( f \circ \pi_\ell \) is a term function of \( M \).

Case 3: \( \alpha(X) = \{1\} \)

As this case is very similar to the previous one, some of the more detailed explanations are omitted.

Let \( s = \langle b_1, a_1 \rangle \). Note \( b_1 \leq a_1 \). Then \( \alpha(s) = \langle 1, 1 \rangle \), and so \( \alpha(s) \not\in R^M_1 \). Thus \( s \not\in R^X_1 \), and hence there is some coordinate \( \ell \) for which \( s(\ell) \not\in R^M_1 \). Therefore there is some \( f \in \text{clone}_u(M) \) with \( f(s(\ell)) = \langle f(b_1(\ell)), f(a_1(\ell)) \rangle = \langle 1, 1 \rangle = \langle \alpha(b_1), \alpha(a_1) \rangle \). Since for every \( x \in X \), \( \alpha(x) = 1 \) by the case assumption, we have \( b_1 \leq x \leq a_1 \), so \( 1 = f(b_1(\ell)) \leq f(x(\ell)) \leq f(a_1(\ell)) = 1 \). Therefore \( f(x(\ell)) = 1 \), and \( \alpha = f \circ \pi_\ell \mid_X \).
Case 4: $\alpha(X) \not\in \{0, 1\}$

Let $S = \alpha(X)$. Recall that we have an enumeration on the elements of $S$ as $\{m_0, m_1, \ldots, m_k\}$ with $m_0 \leq m_1 \leq \ldots \leq m_k$. Let $s = \langle b_{m_0}, a_{m_0}, b_{m_1}, a_{m_1}, \ldots, b_{m_k}, a_{m_k} \rangle$, noting that for each $m_i$, we have $b_{m_i} \leq a_{m_i}$. Then $\alpha(s) = \langle m_0, m_0, m_1, \ldots, m_k, m_k \rangle$, so $\alpha(s) \not\in R^M_S$ and $s \not\in R^X_S$. Thus there exists a coordinate $\ell$ such that $s(\ell) \not\in R^M_S$. That is,

$$s(\ell) = \langle b_{m_0}(\ell), a_{m_0}(\ell), b_{m_1}(\ell), a_{m_1}(\ell), \ldots, b_{m_k}(\ell), a_{m_k}(\ell) \rangle = \langle m_0, m_0, m_1, \ldots, m_k, m_k \rangle.$$

Suppose $x \in \alpha^{-1}(\{m_i\})$. Then $b_{m_i} \leq x \leq a_{m_i}$, so $m_i = b_{m_i}(\ell) \leq x(\ell) \leq a_{m_i}(\ell) = m_i$. Thus $x(\ell) = m_i$, and $\alpha(x) = \pi(\ell)x$. Thus $\alpha$ is the projection map onto the coordinate $\ell$, which extends to the term function of $M$ given by the $\ell$th projection map. □

This concludes the proof that $M$ satisfies the interpolation condition relative to $M$, and hence, $M$ is dualisable.

Recall the statement of Theorem 11:

**Theorem 11.** Let $M$ be a $\{0, 1\}$-valued unary algebra with zero such that $\text{Rows}(M)$ is uniquely witnessed and forms an order ideal under $0 < 1$. Then $M$ is dualisable if and only if $\text{Rows}(M)$ is a lattice order.

Section 4.1 provided the proof in the case that $\text{Rows}(M)$ is not a lattice order, and Section 4.2 provided the proof of the converse. Hence we have concluded the proof of Theorem 11.
Chapter 5

Two-Term Reducts

Algebras for which the rows form an order ideal is a fairly narrow span of \{0, 1\}-valued unary algebras. To what other algebras does Theorem 8 apply? To find a partial answer to this question, we look at two-term reducts. A two-term reduct of an algebra \(P\) has the same underlying set as \(P\), and the fundamental operations are a two element subset of the terms of \(P\). Throughout this section, we consider algebras \(P\) with unique rows, and two-term reducts, \(N\), of \(P\). Note that although the rows of \(P\) are uniquely witnessed, this is not necessarily the case for a reduct \(N\). For any particular element \(a \in P = N\), we need to distinguish which algebra we consider \(a\) to be in; in particular, we denote the row of \(a\) in \(P\) as \(\text{row}^P(a)\) and the row of \(a\) in \(N\) as \(\text{row}^N(a)\).

In this chapter, we conjoin pp-formulae with additional atomic formulae. Throughout this chapter, if we have a pp-formula of the form \(\Phi : \exists w [\theta]\) and we construct a new pp-formula of the form \(\Phi \& f_1(w) \approx f_2(w)\), we mean the formula \(\exists w [\theta \& f_1(w) \approx f_2(w)]\).
5.1 Two-Term Reducts and Non-Dualisability

Suppose $\mathbf{P} = \langle P, F \rangle$ is a $\{0, 1\}$-valued unary algebra with 0 such that $\text{Rows}(\mathbf{P})$ is uniquely witnessed. If $\mathbf{P}$ has a two-term reduct $\mathbf{N}$ such that $\text{Rows}(\mathbf{N}) = \{(0,0), (0,1), (1,0)\}$ (i.e., is the v-order ideal relation) where 0 is the unique witness of $(0,0) \in \text{Rows}(\mathbf{N})$, we say that $\mathbf{P}$ is a v-order reductable algebra and that $\mathbf{N}$ is a v-order reduct of $\mathbf{P}$. In particular, for $n \geq 1$ and $m \geq 1$ we say that $\mathbf{P}$ is an $(n,m)$-reductable algebra if there is a v-order reduct $\mathbf{N}$ or $\mathbf{P}$ with exactly $n$ repetitions of the row $(0,1)$, and exactly $m$ repetitions of the row $(1,0)$. In this case, we say that $\mathbf{N}$ is a $(n,m)$-reduct of $\mathbf{P}$. Note that every v-order reductable algebra must be an $(n,m)$-reductable algebra for some $n$ and $m$, and also that an algebra $\mathbf{P}$ may be both an $(n,m)$- and a $(k,\ell)$-reductable algebra for $(n,m) \neq (k,\ell)$, such as the example in Figure 5.1.

As Theorem 19 makes use of v-ghosting formulae, let us return to the definition:

Let $\mathbf{M}$ be a $\{0,1\}$-valued unary algebra with 0. Suppose there exist $Z \subseteq F_1$, (where $F_1$ is the set of non-constant unary term operations of $\mathbf{M}$) distinct $t$ and $u \in F_1 \setminus Z$, and a possibly empty collection $\{E_i\}$ of subsets of $F_1$ such that we can pp-define the v-order ideal relation $R = \{(0,0), (0,1), (1,0)\}$ via

$$\Phi: \exists w \left[ \bigwedge_{z \in Z} (z(w) \approx 0) \& \left[ \bigwedge_{E \in \{E_i\}} \left[ \bigwedge_{d,e \in E} (d(w) \approx e(w)) \right] \& [x \approx t(w)] \& [y \approx u(w)] \right] \right]$$

(3.1 revisited)

such that if $w_1$ and $w_2$ witness the same element of $R$, then $w_1 = w_2$. Then $\Phi$ is a v-ghosting formula for $\mathbf{M}$.

In the proof of Theorem 19, we construct a v-ghosting formula on an algebra by conjoining a v-ghosting formula on another algebra with additional atomic formulae. Lemma 18 provides a necessary step in demonstrating such a conjunction can be extended to a v-ghosting formula. Using the notation of Lemma 18, we call $\Phi$ an extension of $\Phi^\ast$. 
Lemma 18. Let $M$ be a finite $\{0,1\}$-valued unary algebra with zero, $f_1, f_2 \in F_1$, and 

$$
\Phi^* : \exists w \left[ \bigwedge_{z \in Z^*} (z(w) \equiv 0) \land \left( \bigwedge_{E \in \{E^*_i\}} \left( \bigwedge_{d,e \in E^*_i} d(w) \equiv e(w) \right) \right) \land [x \equiv t(w)] \land [y \equiv u(w)] \right].
$$

Suppose $\Phi^* \land [f_1(w) \equiv f_2(w)]$ $pp$-defines the relation $R$ and $w_r$ is the unique witness to $[\Phi^* \land f_1(w) \equiv f_2(w)] (r)$ for each $r$ in $R$. Then there exists a $pp$-formula $\Phi$ of the form of Formula 3.1 that $pp$-defines the relation $R$ such that $w_r$ is the unique witness of $\Phi (r)$ for each $r$ in $R$.

Proof. Let $K$ be the set of all terms $k \in F_1$ such that $f_1(w) \equiv k(w)$ occurs in $\left( \bigwedge_{E \in \{E^*_i\}} \left( \bigwedge_{d,e \in E^*_i} d(w) \equiv e(w) \right) \right)$ and $L$ be the set of all terms $\ell \in F_1$ such that $f_2(w) \equiv \ell(w)$ occurs in $\left( \bigwedge_{E \in \{E^*_i\}} \left( \bigwedge_{d,e \in E^*_i} d(w) \equiv e(w) \right) \right)$. Let $\Phi = \Phi^* \land [f_1(w) \equiv f_2(w)] \land \left( \bigwedge_{k \in K, \ell \in L} k(w) \equiv \ell(w) \right)$. Since $\Phi$ is a conjunction of $\Phi^* \land [f_1(w) \equiv f_2(w)]$ and additional atomic formulae, any satisfactory witness of $\Phi$ is a satisfactory witness of $\Phi^* \land [f_1(w) \equiv f_2(w)]$. Furthermore, the only occurrence of free variables is in $\Phi^* \land [f_1(w) \equiv f_2(w)]$, so if $w_r$ witnesses $\Phi (r)$, then $w_r$ also witnesses $[\Phi^* \land f_1(w) \equiv f_2(w)] (r)$. It thus suffices to show that every satisfactory witness of $\Phi^* \land [f_1(w) \equiv f_2(w)]$ is a satisfactory witness of $\Phi$. Let $w_r$ be a satisfactory witness of $\Phi^* \land [f_1(w) \equiv f_2(w)]$, and suppose for a contradiction that $w_r$ is not a satisfactory witness of $\Phi$. Then there is a $k \in K$ and $\ell \in L$ such that $k(w_r) \neq \ell(w_r)$. But $w_r$ satisfies $\Phi^* \land [f_1(w) \equiv f_2(w)]$, so $f_1(w_r) = k(w_r)$, $f_2(w_r) = \ell(w_r)$, and $f_1(w_r) = f_2(w_r)$. Thus $k(w_r) = \ell(w_r)$ contradicting $k(w_r) \neq \ell(w_r)$. Hence $w_r$ is a satisfactory witness of $\Phi$. □

We now have the necessary tools to prove the main result of this chapter.

Theorem 19. Every $v$-order reductable algebra $P$ has a $v$-ghosting formula of the form

$$
\exists w \left[ \bigwedge_{z \in Z^*} (z(w) \equiv 0) \land \left( \bigwedge_{E \in \{E^*_i\}} \left( \bigwedge_{d,e \in E^*_i} d(w) \equiv e(w) \right) \right) \land [x \equiv f_1(w)] \land [y \equiv f_2(w)] \right].
$$

where $f_1$ and $f_2$ are the fundamental operations of a $v$-order reduct of $P$. 49
Figure 5.1: $P$ is a v-order reductable algebra that $(1,3)$-reduces to $N_1$ and $(2,2)$-reduces to $N_2$.

**Proof.** Using the fact that every v-order reductable algebra is a $(n,m)$-reductable algebra for some $n$ and $m$, we prove this theorem by performing a double induction on $n$ and $m$.

To establish the base case, suppose $P$ is a $(1,1)$-reductable algebra with a $(1,1)$-reduct $N = \langle P; f_1, f_2 \rangle$. Let $p$ be the unique element with $\text{row}^N(p) = (0,1)$, and $q$ be the unique element with $\text{row}^N(q) = (1,0)$. Note that since $N$ is a v-order reduct, 0 is necessarily the unique element with $\text{row}^N(0) = (0,0)$. Uniqueness implies that $N = \{0, p, q\}$. Recall that a v-ghosting formula is a pp-formula of the form

$$\exists w \left[ \bigwedge_{z \in Z} w(z) \approx 0 \right] \land \left[ \bigwedge_{E \subseteq \{E\}} \bigwedge_{d \in E} d(w) \approx e(w) \right] \land [x \approx t(w)] \land [y \approx u(w)]$$

(where $t$ and $w$ are arbitrary terms) that pp-defines the v-order ideal relation with unique witnesses.

We claim that

$$\exists w [x \approx f_1(w)] \land [y \approx f_2(w)]$$

(5.2)

is a v-ghosting formula for $P$. Indeed, taking $Z = E = \emptyset$, $t = f_1$, and $u = f_2$, Formula 5.2 has the form of a v-ghosting formula. As reducts have the same universe as the algebras from which they are derived, $P = N = \{0, p, q\}$. Then the element 0 witnesses $\langle f_1(0), f_2(0) \rangle = (0,0)$, the element $p$ witnesses $\langle f_1(p), f_2(p) \rangle = (0,1)$, and $q$ witnesses $\langle f_1(q), f_2(q) \rangle = (1,0)$. Hence Formula 5.2
pp-defines the v-order ideal relation, and since $P = \{0, p, q\}$, we must have unique witnesses for this relation. Thus $P$ has a v-ghosting formula of the form of Formula 5.1.

Now assume that every $(1, m)$-reductable algebra is a v-ghostable algebra with a v-ghosting formula of the form of Formula 5.1. Suppose that $\mathbb{P}$ is a $(1, m + 1)$-reductable algebra with a $(1, m + 1)$-reduct $N = \langle P; f_1, f_2 \rangle$. Pick an element $q'$ with row $N(q') = (1, 0)$ and $q' \neq 1$. We may do this since $m + 1 \geq 2$, so $N$ has at least two repetitions of the row $(1, 0)$. By Lemma 2, $P' = P \setminus \{q'\}$ is a subuniverse of $\mathbb{P}$ since $q' \notin \{0, 1\}$. Then the two-term reduct $N' = \langle P'; f_1, f_2 \rangle$ of $\mathbb{P}'$ has exactly $m$ repetitions of the row $(1, 0)$, so $\mathbb{P}'$ is a $(1, m)$-reductable algebra. Hence by the inductive hypothesis, $\mathbb{P}'$ has a v-ghosting formula $\Phi'$ of the form of Formula 5.1. Recall that 0 is always the unique witness of $(0, 0)$ in a v-ghosting formula. Let $p$ and $q$ be the witnesses (in $\mathbb{P}'$) of $\Phi'(0, 1)$ and $\Phi'(1, 0)$ respectively (i.e. $\langle f_1(p), f_2(p) \rangle = (0, 1)$ and $\langle f_1(q), f_2(q) \rangle = (1, 0)$). Then $0$, $p$, and $q$ are also witnesses of $\Phi'$ when considered as elements of $\mathbb{P}$. However, $q'$ may or may not be a satisfactory witness for $\Phi'$. If not, then $\Phi'$ has unique witnesses in $\mathbb{P}$, so $\Phi'$ is a v-ghosting formula for $\mathbb{P}$. If so, as row $N(q') = (1, 0)$, we have $\langle f_1(q'), f_2(q') \rangle = (1, 0)$ so there are precisely two witnesses of $\Phi'(1, 0)$, specifically $q$ and $q'$. Then since $\mathbb{P}$ has uniquely witnessed rows, there exists a term $f_3 \notin \{f_1, f_2\}$ with $f_3(q) \neq f_3(q')$. Let $\{q_0, q_1\} = \{q, q'\}$ so that $f_3(q_0) = 0$ and $f_3(q_1) = 1.$

<table>
<thead>
<tr>
<th>$f_1$</th>
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<th>$f_3$</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$p$</td>
<td>0</td>
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</tr>
<tr>
<td>$q_0$</td>
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<td>0</td>
</tr>
<tr>
<td>$q_1$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 5.2: The behaviour of $0$, $p$, $q_0$, and $q_1$
Figure 5.2 summarizes this information. We have either $f_3(p) = 0$ or $f_3(p) = 1$. If $f_3(p) = 0$, let $\Phi^* = \Phi' \& [f_3(w) \approx 0]$, and if $f_3(p) = 1$, let $\Phi^* = \Phi' \& [f_3(w) \approx f_2(w)]$. In either case, $q_1$ is not a satisfactory witness of $\Phi^*$, but $0$, $p$, and $q_0$ are. By Lemma 18, there is an extension $\Phi$ of $\Phi^*$. Thus $\Phi$ is a v-ghosting formula for $P$ with unique witnesses $0$, $p$, and $q_0$. Furthermore, in either case we are using $[x \approx f_1(w)] \& [y \approx f_2(w)]$ where $f_1$ and $f_2$ are the fundamental operations of $N$, so $P$ has a v-ghosting formula of the form of Formula 5.1.

Now suppose that every $(n, m)$-reductable algebra is a v-ghostable algebra with a v-ghosting formula of the form of Formula 5.1. Suppose that $P$ is a $(n+1, m)$-reductable algebra with a $(n+1, m)$-reduct $N = \langle P; f_1, f_2 \rangle$. Pick an element $p' \neq 1$ with $row^N(p') = \langle 0, 1 \rangle$. We may do so since $n + 1 \geq 2$. Let $P' = \langle P \setminus \{p'\}; F^P \rangle$. Again, $P'$ is an algebra by Lemma 2 as $p' \notin \{0, 1\}$. Then $P'$ has the two-term reduct $N' = \langle P'; f_1, f_2 \rangle$ with exactly $n$ repetitions of the row $\langle 0, 1 \rangle$, so $P'$ is a $(n, m)$-reductable algebra. Hence by the inductive hypothesis, $P'$ has a v-ghosting formula $\Phi'$ of the form of Formula 5.1. Again, 0 is always the unique witness of $\langle 0, 0 \rangle$ in a v-ghosting formula. Let $p$ and $q$ be the witnesses (in $P'$) of $\Phi'(0, 1)$ and $\Phi'(1, 0)$ respectively. If we consider $\Phi'$ on $P$, $p'$ may or may not be a satisfactory witness. If not, then $\Phi'$ has unique witnesses in $P$, so $\Phi'$ is a v-ghosting formula for $P$. Now assume that $p'$ is a satisfactory witness of $\Phi'$. Similarly as before,
\( P \) has precisely two witnesses for \( \Phi'(0,1) \), namely \( p \) and \( p' \). Since \( P \) has uniquely witnessed rows, there exists a term \( f_3 \notin \{ f_1, f_2 \} \) with \( f_3(p) \neq f_3(p') \). Let \( \{ p_0, p_1 \} = \{ p, p' \} \) so that \( f_3(p_0) = 0 \) and \( f_3(p_1) = 1 \). See Figure 5.3. Then we have either \( f_3(q) = 0 \) or \( f_3(q) = 1 \). If \( f_3(q) = 0 \), let \( \Phi^* = \Phi' \land [f_3(w) \approx 0] \), and if \( f_3(q) = 1 \), let \( \Phi^* = \Phi' \land [f_3(w) \approx f_1(w)] \). In either case, \( p_1 \) is not a witness of \( \Phi^* \), but \( 0, p_0 \), and \( q \) are. By Lemma 18, there is an extension \( \Phi \) of \( \Phi^* \). Thus \( \Phi \) is a \( v \)-ghosting formula for \( P \) with unique witnesses \( 0, p_0 \), and \( q \). Furthermore, in either case we are using \( [x \approx f_1(w)] \land [y \approx f_2(w)] \) where \( f_1 \) and \( f_2 \) are the terms of the \( N \), so \( P \) has a \( v \)-ghosting formula of the form of Formula 5.1.

Thus every \( v \)-order reductable algebra has a \( v \)-ghosting formula of the form of Formula 5.1. \( \square \)

Recall that having a \( v \)-ghosting formula is equivalent to being \( v \)-ghostable. By Theorem 8, we have that every \( v \)-ghostable algebra is not dualisable. This gives the following corollary:

**Corollary 20.** Every \( v \)-order reductable algebra is not dualisable.
Chapter 6

Concluding Notes

6.1 Examples

In this section, we examine a few examples that demonstrate how Theorems 8, 11, and Corollary 20 relate to one another. We repeat these theorems and the necessary definitions here:

Let $\mathbf{M}$ be a $\{0,1\}$-valued unary algebra with 0 with non-constant fundamental operations $F_i$. Suppose there exist $Z \subseteq F_i$, distinct $t$ and $u \in F_i \setminus Z$, and a possibly empty collection $\{E_i\}$ of subsets of $F_i$ such that we can pp-define the $v$-order ideal relation $R = \{(0,0), (0,1), (1,0)\}$ via

$$\Phi: \exists w \left[ \bigwedge_{z \in Z} z(w) \approx 0 \land \bigwedge_{E \in \{E_i\}} \bigwedge_{d,e \in E} d(w) \approx e(w) \land x \approx t(w) \land y \approx u(w) \right]$$

such that if $w_1$ and $w_2$ witness the same element of $R$, then $w_1 = w_2$. Then we say $\mathbf{M}$ is a $v$-ghostable algebra. We also say that $\Phi$ is a $v$-ghosting formula for $\mathbf{M}$.

Theorem 8. $v$-ghostable algebras are not dualisable.

Theorem 11. Let $\mathbf{M}$ be a $\{0,1\}$-valued unary algebra with zero such that $\text{Rows}(\mathbf{M})$ is uniquely witnessed and forms an order ideal under $0 < 1$. Then $\mathbf{M}$ is dualisable if and only if $\text{Rows}(\mathbf{M})$ is a lattice order.
Suppose $P = \langle P, F \rangle$ is a $\{0, 1\}$-valued unary algebra with 0 such that $\text{Rows}(P)$ is uniquely witnessed. If $P$ has a two-term reduct $N$ such that $\text{Rows}(N) = \{(0, 0), (0, 1), (1, 0)\}$ (i.e., is the v-order ideal relation) where 0 is the unique witness of $(0, 0) \in \text{Rows}(N)$, we say that $P$ is a v-order reductable algebra and that $N$ is a v-order reduct of $P$.

**Corollary 20.** Every v-order reductable algebra is not dualisable.

Note that the proofs of Theorem 11 and Corollary 20 use Theorem 8 to prove non-dualisability. Hence for a given algebra $M$ that is not dualisable, if either Theorem 11 or Corollary 20 apply, then so does Theorem 8.

An important difference between Theorem 8 and the other results of this thesis is that Theorem 8 allows for repeated rows. Indeed, consider the algebra $M_0$ in Figure 6.1. Note that $\text{row}(3) = \text{row}(4) = (1, 1, 0)$, so neither Theorem 11 nor Corollary 20 apply as they require unique rows. Consider the pp-formula

$$\Phi : \exists w \ [f_1(w) \approx 0 \& x \approx f_2(w) \& y \approx f_3(w)].$$

This formula pp-defines the v-order ideal relation $\{(0, 0), (0, 1), (1, 0)\}$ with unique witnesses, so $M_0$ is v-ghostable and not dualisable by Theorem 8.

It should also be noted that we can also find examples of algebras with unique rows where Theorem 8 applies, but Theorem 11 and Corollary 20 do not. The algebra $M_1$ in Figure 6.2 is such an algebra. The rows of $M_1$ do not form an order ideal, as the row $\langle 1, 1, 1 \rangle$ is present, but $\langle 1, 0, 1 \rangle$ is not, so Theorem 11 does not apply. The row $\langle 1, 1, 1 \rangle$ forces any two-term reduct of $M_1$ to have $\langle 1, 1 \rangle$ as a row, so Corollary 20 also does not apply. The pp-formula used for $M_0$ provides a v-ghosting formula for $M_1$ as well, so Theorem 8 applies to $M_1$.

Corollary 20 has the obvious limitation that it does not apply when every two-term reduct $N$ has $\langle 0, 0 \rangle$ as a repeated row. This limitation can be used to find an algebra that fits the hypotheses...
Figure 6.1: An algebra with repeated rows to which Theorem 8 applies.

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<thead>
<tr>
<th>M₀</th>
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<tbody>
<tr>
<td>0</td>
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<tr>
<td>1</td>
<td>0 0 1 1</td>
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</tr>
<tr>
<td>4</td>
<td>1 1 1 1</td>
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</table>

Figure 6.2: An algebra with unique rows to which Theorem 8 applies.

<table>
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<tr>
<th>M₁</th>
<th>f₁ f₂ f₃</th>
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<tbody>
<tr>
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<td>3</td>
<td>1 1 0 1</td>
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<tr>
<td>4</td>
<td>1 1 1 1</td>
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of Theorem 11 but not Corollary 20. The algebra M₂ in Figure 6.3 has rows that form an order ideal, but every two-term reduct contains ⟨0, 0⟩ as a repeated row. Hence Theorem 11 applies, but Corollary 20 does not.

Conversely, the algebra M₃ in Figure 6.4 satisfies the hypotheses of Corollary 20 but not Theorem 11. Indeed, N = ⟨M₃; f₁, f₂⟩ is a v-order reduct for M₃. The rows of M₃ are unique but do not form an order ideal, however, as the row ⟨0, 1, 1⟩ is present but ⟨0, 1, 0⟩ is not.

The last example we look at is M₄ in Figure 6.5, which none of the results of this thesis is capable of categorizing. Every two-term reduct of M₄ has ⟨1, 1⟩ as a row, so we cannot use Corollary 20. It is also apparent that the rows do not form an order ideal, so Theorem 11 does not apply. That Theorem 8 does not apply requires a slightly lengthier consideration. Note that any pp-formula that contains an atom of the form \( f_i(w) \approx f_j(w) \) or \( f_i(w) \approx 0 \) for \( i \neq j \) has at most two satisfactory witnesses (for example, \( f_1(w) \approx f_3(w) \) has only 0 and 2 as satisfactory witnesses).

Because v-ghosting formulæ must have exactly three satisfactory witnesses, atoms of these forms cannot appear in any v-ghosting formula for M₄. Hence a v-ghosting formula for M₄ must be of
the form
\[ \exists w \ [ x \approx f_i(w) \& y \approx f_j(w) ] . \]

But every element of \( M_4 \) is a satisfactory witness for formulae of this form. As \( M_4 \) contains four elements and \( v \)-ghosting formulae require exactly three satisfactory witnesses, there can be no \( v \)-ghosting formula for \( M_4 \). Thus Theorem 8 does not apply.

The author, Jennifer Hyndman, and Ross Willard have managed to determine that \( M_4 \) is not dualisable in a personal correspondence using techniques outside the scope of this thesis.

### 6.2 Further Research

The results of this work provide a small step in answering the much larger question of “which unary algebras are dualisable?” There are a few options for a next step to take with this research. One option is to look at algebras with repeated rows. Some preliminary work has been done by Jennifer Hyndman and the author of this thesis in looking at \( \{0, 1\} \)-valued unary algebras with zero much like the ones in this work, except that the row of zeros is a repeated row. The results so far

<table>
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</table>

Figure 6.3: An algebra to which Theorem 11 applies, but Corollary 20 does not.

Figure 6.4: An algebra to which Corollary 20 applies, but Theorem 11 does not.
Figure 6.5: An algebra to which none of our results applies.

<table>
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<tr>
<td>3</td>
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<td>1</td>
<td>0</td>
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</tbody>
</table>

The difference between these two algebras is that the dualisable one has the zero row repeated.

Continuing in the vein of relaxing the hypotheses of our results, it would also be interesting to investigate \( \{0, 1\} \)-valued unary algebras that do not have a constant \( 0 \)-valued function. Because of the heavy dependence on having a constant valued function (particularly in the proof of the $v$-ghosting theorem), a different approach is necessary in classifying these algebras.

One final place to find inspiration for further research is to return to the beginning, and look at the result from [4] that catalysed this thesis:

**Theorem 21.** ([4]) If $M$ is a \( \{0, 1\} \)-valued unary algebra with 0, then one of the following holds:

1. the $\leq$ relation on \( \{0, 1\} \) can be positive primitively defined;
2. the graph of addition modulo 2 on \( \{0, 1\} \) can be positive primitively defined;
3. the rows of $M$ form an order ideal.

In the first two cases, there is no finite basis for the quasi-equations, and in the last case, there is a
finite basis for the quasi-equations.

Theorem 11 provides categorization of dualisability in the third case when the rows of $M$ are uniquely witnessed. The first two cases could perhaps provide a framework in which to find other dualisability results.
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