SPECKLE PATTERN BASED SINGLE PIXEL IMAGING

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Abstract

This thesis introduces a new modality for terahertz single pixel imaging which takes advantage of compressive sensing techniques and random speckle patterns. This proposed modality offers a new trade-off in complexity and speed in comparison to current imaging systems with the introduction of a spatial light modulator that is minimally simple and inexpensive. Experimental results are obtained exhibiting the successful application of this technique. Extensions and applications of the proposed technique are also discussed.
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Introduction

Terahertz (THz) imaging remains an advancing field with broad applications to non-destructive evaluation, and security screening. However, in contrast to the ubiquity of silicon-based sensor technology for visible light in today’s electronics, multi-pixel detectors for wavelengths outside of the sensitive range for silicon remain coarse-grained, noisy, and expensive. In view of these current limitations, an efficient approach called single pixel imaging has recently emerged in which images are collected using only a single detector of the required type without the use of raster scanning. Combined with the mathematical theory of compressive sensing (CS), single pixel imaging now represents a promising technology for imaging far outside of the spectral range of conventional digital cameras.

In single pixel imaging, extended linear samples of a scene under view are optically computed by spatially modulating an imaging beam which either impinges onto or emerges from an imaging target. The linear samples are realized by a wide-aperture detector which integrates the total structured light reflected or transmitted from the scene. Single pixel images are then computed from the sampled collection according to reconstruction algorithms which may be tailored to the particular imaging scenario at hand.Remarkably, although it is required that the spatial structure imposed onto the imaging beam be known with high precision, image reconstruction is possible with completely random modulation patterns in the context of CS theory. Additionally, reconstruction algorithms based on CS are highly efficient, requiring a number of
measurements equal to only a fraction of the image resolution.

Historically, the field of THz single pixel imaging began almost immediately after the first CS-based single pixel camera was demonstrated in 2006 for visible light using an off-the-shelf computer controlled digital micro-mirror device (DMD) for spatial modulation [1]. For THz however, no comparable ‘off-the-shelf’ solution for spatial light modulation currently exists. Although the ‘pixel level’ spatial control of THz beams is an area of active research in the field of metamaterials [2, 3], THz modulators remain cumbersome, complex, and expensive. Despite nearly a decade of research in the field, the main technical challenge for THz single pixel imaging remains the ability to produce detailed and repeatable modulation patterns at high speed.

On the other hand, the utility of random linear sampling in the CS theory suggests that highly random speckle patterns are suitable for THz single pixel imaging. In this case, a natural method exists with which to modulate THz beams: one simply introduces an optically rough and reflective surface to the imaging beam and utilizes the THz speckle pattern produced in the far field. Although in practice such speckle patterns would have to be determined experimentally, a large sequence of such patterns could be readily obtained by translating the reflective surface.

This thesis proposes exactly such a modulation scheme for THz single pixel imaging. By using speckle patterns as the basis for image sampling, ‘pixel level’ control of the THz field can be traded for the experimental mapping of far field speckle patterns produced by a minimally simple and inexpensive modulator. Surprisingly, this trade-off has not yet been explored despite its feasibility for THz radiation. In particular, the millimeter-scale wavelengths of THz imply relatively low tolerances for the mechanical positioning needs of modulator translation and speckle field mapping. With sufficient mechanical repeatability, the mapping of the speckle fields need only be carried out once. In addition, such speckle pattern modulators for THz frequen-
cies are extremely simple to construct: the experiments described in this work used a modulator made by hand using household cardboard and aluminum foil.

**Organization of this Thesis**

**Chapter 1** is dedicated to a description of THz radiation and its broad applications. A demonstration of diffraction effects at THz frequencies is also provided, as is the relevant background on near field and far field imaging, and speckle patterns.

**Chapter 2** contains the relevant background on the mathematical theory of compressive sensing (CS) which is used as the basis of the image reconstruction algorithms used in this work. Two reconstruction methods are described and motivated which are based on the sparsity of the discrete cosine transform (minDCT), and the minimization of the total image variation (minTV).

**Chapter 3** introduces the single pixel imaging technique in detail, including a survey of recent work inside and outside of THz science. The active and passive modes for single pixel imaging are defined. The proposed speckle pattern modulation scheme is also described, along with a comparison to previous single pixel imaging systems.

**Chapter 4** documents the first experimental demonstration of single pixel imaging using speckle pattern modulation. Successful image reconstructions are shown to be possible from sampling rates as low as 10% compared to the image pixel count.

**Chapter 5** finally provides a summary and discussion of the experimental results, and the conclusion of this work.
Three supporting appendices are also included following Chapter 5:

Appendix A provides a further discussion of the discrete cosine transform, including an explicit derivation of the transform equations and a demonstration of transform compression.

Appendix B describes the statistical method by which error parameters needed for the reconstruction algorithms were estimated in the presence of experimental noise. The method presented in this section is a result of the author’s own research.

Appendix C is an extensive gallery of the experimental imaging results obtained for this thesis.
Part I

Theoretical Background
Chapter 1

Terahertz Radiation

Terahertz (THz) radiation lies on the electromagnetic spectrum (figure 1.1) between the microwave and infrared regions, corresponding to a range of frequencies with approximately sub-millimeter wavelengths, from several tens of GHz ($10^9$ Hz) up to several tens of THz ($10^{12}$ Hz). Although it is named according to the metric prefix *tera* (T), the THz region can be more appropriately defined in terms of the unique technologies required for its generation and detection [4]. Conventional optical technologies are unable to efficiently generate frequencies below the tens of THz. On the low end of the range, the generation and detection of radiation approaching the 100 GHz range becomes increasingly difficult with electronic technologies. For many years, a technological ‘THz gap’ persisted representing a difficult frontier in photonics research. Since the 1970s, a multitude of subsequent improvements have now moved the THz range beyond the experimental frontier. Today, with an increasing array of commercial systems and experimental techniques, THz science is routinely conducted in laboratories across the world spanning many applications including material identification [5], security screening [6], and even art preservation [7].

Several attractive properties of THz radiation have also spurred a continuing search for THz applications in industry. For example, THz is transparent to many dry,
non-conductive materials including common packaging materials such as plastics and cardboard. The non-ionizing and sub-millimeter wavelength characteristics of THz also make it an alternative to X-Ray imaging, providing sufficient resolution for most imaging applications without the associated health risks.

Broadly speaking, two kinds of THz apparatus are found in the laboratory. First, are coherent or continuous wave (cw) THz sources which produce monochromatic sub-millimeter beams, and are often coupled with intensity-based photodetectors. Second, more dominant in the literature, are pulsed laser-driven THz sources which produce broadband THz field transients coupled with pulsed detection schemes allowing for explicit time-dependent measurements of the electric field [8]. Although commercial realizations of both systems now exist for the research community, current THz sensing equipment remains subject to ongoing improvements in both cost and complexity.

Of course, a comprehensive review of the applications for THz radiation would extend well beyond the scope of this thesis. Fortunately, excellent review articles for recent work in THz imaging and spectroscopy can be found in the literature by Chan and Mittleman [9], and by Baxter and Guglietta [10]. Nonetheless, to illustrate the diversity of THz science, a few recent studies are mentioned below.
A Survey of THz Science

A growing variety of imaging, spectroscopy, and interferometry applications for cw THz is found in the literature. A tunable cw THz source was used by Sun et al. to spectroscopically differentiate two isotopic variants of carbon monoxide [11]. Single-frequency THz sources have also been used for a variety of coherent imaging applications with counterparts in the visible range. An experiment by Cull et al. demonstrated a technique for digital on-axis holography using a THz source [12]. Quite recently, the measurement of wood index and absorption properties using a Michelson interferometer was presented in a thesis by Lawyer [13]. A compact single-frequency system operating at 0.2 THz was presented by Karpowicz et al. and used to detect weapons through a leather briefcase, and locate defects in insulating panels from the NASA space shuttle [14].

The technique of THz time-domain imaging has also represented an important application in the THz field. In THz time-domain spectroscopy (THz-TDS), Fourier transforms of broadband THz pulses are used to extract the phase-sensitive dielectric properties of materials. The application of THz-TDS for the study and identification of polar molecular gas species is several decades old [15]. By measuring the time-of-flight delay for THz pulses in reflection, depth information can be obtained for hidden surfaces. Using this principal, a system for THz reflection tomography was presented by Jin et al. who obtained detailed 3D reconstructions of the composite layers of a floppy disk [16]. A pulsed THz reflection imaging apparatus was also used by Jackson et al. to image growth-rings in wood hidden beneath layers of varnish and paint [17]. By combining THz-TDS techniques with raster scanning of images, it is possible to obtain multi-spectral images which can be viewed at any wavelength within the spectral content of the probing THz pulse. A study by Fukunaga et al. used this technique along with THz spectral data to chemically distinguish ink pigments
from stains on a 13th century parchment [18].

Although the majority of current THz applications have been realized with either cw or pulsed THz systems, specialized near room temperature black-body radiators have also been used as sources of incoherent THz radiation for experiments in passive millimeter wave imaging (PMMWI) for security applications [19].

Finally, two THz studies have been presented by the author. In the first study, a pulsed THz system was used to study the pressure scaling of the rotational spectra of carbon monoxide (CO) using THz-TDS [20]. In the second study, another pulsed time-domain THz system was used to conduct Brewster angle reflection imaging of concealed water filled voids [21].

1.0.1 Diffraction Effects

The behaviour of light which is incident on wavelength sized objects is not consistent with the familiar geometrical ‘ray model’ of light propagation. In such cases, the wave nature of light gives rise to complex diffraction phenomena, in which a fraction of the propagating light is dispersed by object edges into diverging waves which propagate and interfere with the entire continuous wavefront. Diffraction effects also fundamentally limit the minimum spot size to which light can be focused using lenses, which roughly speaking is determined by the wavelength multiplied by the $f$-number of the focusing optics [22]. The resolution of conventional imaging systems is therefore strongly determined by the diffraction limit, with the smallest resolvable features in an image being not generally smaller than the wavelength of the imaging radiation [9].

Due to its macroscopic wavelength, diffraction universally influences the behaviour of THz radiation in table top experiments. The distorted transmission of pulsed THz light by a sub-wavelength circular aperture of decreasing size has been experimentally studied by Mitrofanaov et al. [23]. As an example of this effect, a series of diffracted
Figure 1.2: Experimental example of THz light diffraction in the near field. Eleven images (16 by 16 pixels) are taken in 4 mm backwards offsets over 4.0 cm (left to right) of an ‘H’ shaped aperture (1.7 cm height) at 102 GHz ($\lambda = 2.9$ mm). A photograph of the aperture is shown in the circle at left. The THz source and camera used to collect each image are described later in Chapter 4. Due to the thickness of an enclosure panel covering the sensor array, the first image was separated from the sensor plane by an optical path length of approximately 3.5 mm.

images of a custom aperture is shown in figure 1.2, which illustrates the complex structure caused by diffraction imposed in the propagating field.

Near Field and Far Field Diffraction

Although a general mathematical treatment of diffraction effects will not be required for the presentation of this thesis, it will be useful to make a qualitative distinction between the near field and far field behaviour of diffracted light. Broadly speaking, the diffraction light incident on an aperture may be considered as a fracturing of the beam into two co-interfering components: set of orders of propagating transmitted waves consisting mainly of unaffected light passing freely through an aperture area, and a set of evanescent waves which interact with optically rough edge features and dissipate quickly in space [9]. At close distances to the aperture, i.e. in the near field, the diffracted scalar field exhibits a microscopic structure component which is progressively attenuated at increasing distances, i.e. in the far field. The loss of this information to detectors at far field distances is an alternate way in which to regard the diffraction limited optics of lenses.
Mathematically, the theory of scalar wave diffraction described by the Fresnel-Kirchhoff formula [22, 24] is neatly partitioned by two integral approximation regimes for light of wavelength $\lambda$ propagated to distances $d$ from apertures of area $A$: the case for the near field in which $d < \frac{A}{\lambda}$ (Fresnel diffraction), and the case for the far field which $d \gg \frac{A}{\lambda}$ (Fraunhofer diffraction). For a monochromatic THz beam of wavelength $\lambda = 3$ mm incident on an aperture with a centimeter squared area, the near and far field diffraction regimes split roughly into the cases $d < 3$ cm and $d \gg 3$ cm, respectively.

It is important to note that sub-wavelength spatial resolutions can be achieved if light is recorded at near field distances from objects [9]. Sub-wavelength resolutions may be achieved for example by fitting a sufficiently small sub-wavelength aperture onto a photosensor and scanning across the region of interest. In this way, imaging of a light intensity pattern by scanning directly at the region of interest is possible with spatial resolutions determined only by the restrictions on aperture thickness and size and detector sensitivity. Diffraction effects however ultimately influence the transmission behaviour of light through such small apertures. Nonlinear scaling of transmitted energy with aperture area has been observed for pulsed THz radiation by Mitrofanov et al. through apertures as small as $\lambda/300$ [23].

**Speckle Patterns**

Although the previous discussion has centered around the diffraction of optically rough objects and apertures, an entirely new qualitative phenomena is encountered when coherent light encounters extended surfaces with random sub-wavelength scale features. The resulting diffraction effects, enhanced by the surface’s random topography and extended area, are collectively known as *speckle patterns* which are characterized by their extremely complex and random illumination structure. Speckle patterns are commonly observed by eye when an expanded visible laser beam is used to illuminate
flat unpolished surfaces leading to a fine-grained ‘speckled’ appearance. In terms of a scalar field, points in the diffracted speckle pattern are determined by a superposition of field phasors continuously summed across the entire illuminated surface. If the dimensions of the illuminated area are not large compared to the observer distance in the speckle pattern, the field vectors sum in the manner of a ‘3D random walk’, having approximately equal amplitude but highly random phases.

The size of the speckle pattern ‘grains’ is determined by the wavelength, illuminated surface area, and distance to the diffracting surface. As explained by Jones and Wykes [25], the coherent speckle size $s$ can be determined from an autocorrelation function derived from the statistics of the speckle field. The result for speckle size at a distance $d$ from an $L \times L$ surface area illuminated by monochromatic light of wavelength $\lambda$ is

$$s = \frac{\lambda d}{L}. \quad (1.1)$$

When speckle patterns are imaged by lenses, the resulting focal plane distributions resemble speckle patterns but with differing grain size. A distinction can therefore be made between objective speckle, in which the speckle distribution and grain size are determined solely from the plane area and observation distance, and subjective speckle which is determined by the aperture size of the focusing optics [25].

The proposed technique of speckle pattern based single pixel imaging uses optically rough reflectors to create random objective speckle patterns as a means for optically computing random linear samples. Fortunately for THz wavelengths, such reflective surfaces with sub-wavelength texture features are trivial to construct, for example by crumpling a piece of metal foil. Moreover, the scanned near field measurement of THz speckle patterns is also mechanically feasible without high cost. Together, these two observations form the basis for the proposed technique as an alternative approach to THz single pixel imaging. Following the background on compressive sensing presented
in the next chapter, a full description of single pixel imaging, and the proposed speckle pattern based imaging, will be given in Chapter 3.
Chapter 2

Compressive Sensing

Compressive sensing (CS), also known as compressed sensing or compressive sampling, is an alternate approach to signal sampling and reconstruction that in many cases greatly improves upon the theoretical bounds of the classic Shannon/Nyquist sampling theorem [26, 27]: that the perfect recovery of a signal with a finite bandwidth is guaranteed if the sampling rate exceeds the rate of twice the maximum frequency component of the signal. Compressive sensing is a recent theory which, for certain classes of signals, offers vast improvements over the sampling constraints of the Shannon/Nyquist theorem while still achieving perfect or near-perfect reconstruction quality.

In essence, CS deals with classes of sparse signals, that is, classes of signals which have a large number of zero-valued coordinates when represented using a predetermined basis. This additional constraint is what enables the results of CS to improve upon the Shannon/Nyquist result. Traditional methods of signal sensing often suffer from oversampling, relying instead on compression techniques performed after signal acquisition in order to store the signals. An example of this is found in high-resolution digital cameras, whose raw photos often involve many megapixels of image data which are then compressed using various methods in order to be stored.
Compressive sensing operates in a way which may be regarded as combining the signal acquisition and subsequent compression steps into a single non-adaptive step which takes advantage of the underlying sparse nature of the class of signals.

The surprising and non-adaptive nature of this sampling technique is due to the fact that compressive sensing uses linear sampling functions to span the entire signal in fewer samples. In practice, once a small number of such linear samples are obtained, any number of various mathematical reconstruction techniques then take over, using the assumed sparsity of the signal to determine a solution matching the collected data when the linear sampling function is applied. This property of combining the acquisition of signals and their subsequent compression is what underlies the name compressive sensing.

2.1 Overview of Compressive Sensing

We consider real-valued, discrete signals of finite length $N$ as vectors $x \in \mathbb{R}^N$. Higher dimensional signals, such as 2D images, can always be treated in this way by specifying a lexicographic ordering of elements. In compressive sensing, a sample or measurement $y \in \mathbb{R}$ of the signal $x$ is obtained as a linear combination of the signal elements as specified by a known measurement vector $\phi \in \mathbb{R}^N$,

$$y = \sum_{i=1}^{N} \phi_i x_i = \langle \phi, x \rangle .$$  \hspace{1cm} (2.1)

A sequence of $M$ measurements $y_i = \langle \phi_i, x \rangle$ may therefore be specified by a measurement matrix $\Phi \in \mathbb{R}^{M \times N}$ whose rows are formed by the measurement vectors $\phi_i$ for $1 \leq i \leq M$. Writing this in matrix-vector notation yields

$$y = \Phi x ,$$  \hspace{1cm} (2.2)
for the vector of measurements $y \in \mathbb{R}^M$. In the case that $M \geq N$, it is clear that the signal $x$ can be obtained from equation (2.2) uniquely if the $M$ rows of $\Phi$ span $\mathbb{R}^N$. However, for the case that $M \leq N$, there are infinitely many solution signals which exist and are consistent with the observations $y$ via equation (2.2).

In practice, it is strongly desired to minimize the number of measurements even towards the case where $M \ll N$. To proceed with solving equation (2.2), additional constraints must be placed on our signals $x$. Let $\{\psi_i\}_{i=1}^N$ be a pre-determined orthonormal basis for the space $\mathbb{R}^N$. We may then express every vector $x$ in the space as a unique linear combination of the basis vectors:

$$x = \sum_{i=1}^N z_i \psi_i,$$

(2.3)

where the coefficients $z_i$ are determined by the inner products $z_i = \langle x, \psi_i \rangle$. Let $\Psi \in \mathbb{R}^{N \times N}$ be the matrix whose $i$-th column is the $i$-th basis vector $\psi_i$. We may then represent $x$ in matrix-vector notation as $x = \Psi z$, where the matrix $\Psi$ is orthogonal with inverse $\Psi^T$. The vector $z$ therefore constitutes an equivalent representation of the signal $x$. Substituting this into the measurement matrix equation (2.2), we obtain

$$y = \Phi \Psi z = \Theta z,$$

(2.4)

where the $M \times N$ matrix $\Theta$ is defined by the matrix product $\Theta = \Phi \Psi$.

We now obtain the desired constraint on equation (2.2) by assuming that our signal $x$ is $k$-sparse with respect to the chosen orthonormal basis $\Psi$. By this we mean that $x$ is a linear combination of only $k \ll N$ basis vectors, or equivalently that the associated coordinate vector $z$ has precisely $k \ll N$ non-zero coordinates. With these constraints at hand, the computation of candidate solutions to the general measurement matrix equation $y = \Phi x$ now becomes feasible, although in general,
Compressive sensing, as a topic in computational mathematics, now emerges as the body of results describing efficient reconstruction algorithms and the design of measurement matrices \( \Phi \), such that k-sparse signals may be successfully recovered with overwhelming probability from a minimal number of measurements \( M \approx k \).

The application of CS methods to the acquisition of real signals has led to some intriguing interpretations of the technique. With the sparsity-basis \( \Psi \) and measurement matrix \( \Phi \) specified, it is merely necessary to collect a small number of measurements \( M \ll N \) from which the signal \( x \) may be recovered at a later time using an appropriate reconstruction algorithm. The measurements \( y \) may therefore be construed as a compressed representation of the original signal, able to be efficiently transmitted and stored. Remarkably, we see that the compression and the acquisition of the signal \( x \) via the measurements \( y \) occurs simultaneously at the time of measurement. This may be easily contrasted to more conventional acquisition systems such as CCD cameras, in which the entire pixel information of an image must be acquired, transmitted, and stored before compression can take place at the necessary expense of computational power.

### 2.1.1 The First Program - minDCT

The notion of a signal \( x \) being \( k \)-sparse with respect to a basis \( \Psi \) can also be equivalently described using the \( \ell_0 \)-"norm" \( \|z\|_0 \) of the transformed signal\(^1\). Therefore, the most intuitive program for reconstructing the signal \( x \) from the compressive samples

\(^1\)The \( \ell_0 \)-"norm" \( \|x\|_0 \) is defined as the number of non-zero coordinates of \( x \), and is equivalent to the vector’s Hamming distance from zero. The use of quotations is common, and reflects the fact that although \( \| \cdot \|_0 \) satisfies the triangle inequality, it is not truly a norm since it does not satisfy \( \|az\| = |a|\|z\| \) in general for scalars \( a \in \mathbb{R} \). It should be noted that in describing the properties of \( \| \cdot \|_0 \), some authors have mistakenly claimed that it does not satisfy the triangle inequality.
y might be expressed as the solution \( \hat{z} \) to the following optimization problem:

\[
(P_0) \quad \hat{z} = \arg \min \|z\|_0 \quad \text{subject to} \quad \Theta z = y.
\]

Unfortunately, and despite its reasonable motivation, solutions to \( P_0 \) are computationally intensive, requiring an exhaustive combinatorial search of every subset of columns from \( \Theta \). Although there exist conditions on the matrix \( \Theta \) which guarantee uniqueness [28], for most realistic sampling problems, solving \( P_0 \) is simply infeasible.

A breakthrough was the discovery that a computationally feasible yet accurate relaxation of \( P_0 \) could be obtained by substituting the \( \ell_1 \)-norm\(^2\) into the same program [29]:

\[
(P_1) \quad \hat{z} = \arg \min \|z\|_1 \quad \text{subject to} \quad \Theta z = y.
\]

This relaxation, known as basis pursuit, was shown to produce an equivalent solution to \( P_0 \) with large probability, while being amenable to well established and efficient algorithms in linear programming. A wealth of subsequent results soon followed, revealing other programs based on the \( \ell_1 \)-norm to be useful reconstruction algorithms.

This \( \ell_1 \)-norm relaxation, for this thesis, culminates in the following program capable of finding sparse solutions even in the case of random noise present in the measurements:

\[
(P_2) \quad \hat{z} = \arg \min \|z\|_1 \quad \text{subject to} \quad \|\Theta z - y\|_2 \leq \epsilon.
\]

The above program, amenable to established methods in convex optimization, constitutes one of the two reconstruction methods used in this work. It is also known as basis pursuit with inequality constraints [28]. The parameter \( \epsilon \) in \( P_2 \) serves as an error tolerance parameter, determining the maximum acceptable level of disagreement between

\(^2\)The \( \ell_1 \)-norm is defined as \( \|x\|_1 = \sum_{i=1}^{N} |x_i| \).
the candidate reconstruction and the observations in terms of the usual euclidean norm $\| \cdot \|_2$. It will be of interest later to estimate this sole reconstruction parameter in the context of the experiment. We shall however withhold such a discussion until it is necessary.

In addition to an error parameter $\epsilon$, we must of course specify the \textit{sparsity basis} $\Psi$ in order to fully determine $\Theta$ and recover $x$ from $z$. We shall use the discrete cosine transform (DCT) basis for all our reconstructions based on $P_2$, referring to the algorithm as minDCT from now on. The DCT basis is widely known for its ability to compress real-world image data, and was used as the underlying transform involved in the original JPEG image compression standard [30]. For reference, an explicit derivation and discussion of the DCT basis is provided in Appendix A.

### 2.1.2 The Second Program - minTV

When the underlying signal $x$ represents a two dimensional image, an alternate sparsity model may be used to recover the signal from the measurements $y = \Phi x$. Here, instead of assuming sparsity with respect to a given basis, the signal is assumed to be sparse with respect to a \textit{discrete gradient} measured by the \textit{total variation} [29].

Suppose that the signal $x \in \mathbb{R}^N$ is represented as a $K$ by $L$ matrix where $KL = N$ with coordinates (row, column) denoted by $x_{i,j}$, with $i$ and $j$ both ranging starting from 1. The total variation of $x$ is then defined as

$$TV(x) \equiv \sum_{i=1}^{K-1} \sum_{j=1}^{L-1} \sqrt{(x_{i+1,j} - x_{i,j})^2 + (x_{i,j+1} - x_{i,j})^2}.$$  \hfill (2.5)

The result is the following program for the reconstruction of 2D images:

$$(\min TV) \quad \hat{x} = \arg \min TV(x) \quad \text{subject to} \quad \|\Phi x - y\|_2 \leq \epsilon.$$
The solution \( \hat{x} \) to (minTV) therefore approximates the original signal \( x \) without the need to convert the signal from a sparsity basis. The solution tolerance parameter \( \varepsilon \) may be regarded in exactly the same way as in \( P_2 \).

Solutions which minimize the total variation may be interpreted as follows. If a discrete gradient \( \nabla x_{i,j} \) is introduced as the 2-vector

\[
\nabla x_{i,j} = \begin{pmatrix} x_{i+1,j} - x_{i,j} \\ x_{i,j+1} - x_{i,j} \end{pmatrix},
\]

then the total variation may be written simply the sum of the magnitudes of the gradient at each point:

\[
TV(x) = \sum_{i=1}^{K-1} \sum_{j=1}^{L-1} \| \nabla x_{i,j} \|_2.
\]

It is clear that the discrete gradient takes on non-zero values at the site of sharp contrasts between adjacent pixel values in either the vertical or horizontal directions. The magnitude of the discrete gradient is therefore large at the site of edges, with the total variation taking on large values for images with high edge counts or oscillations. Images which feature noise or other spurious high frequency content are therefore avoided by minTV in favor of those images which are spatially contiguous, and which satisfy the observations \( y = \Phi x \) to within the specified tolerance.

Historically, image reconstruction problems similar to minTV arose some decades before compressive sensing in the context of computational noise removal. An early method proposed by Rudin et al. \[31\] is in fact equivalent to minTV in the case that \( \Phi \) is equal to the \( N \) by \( N \) identity matrix. The minTV program may therefore be viewed as an adaptation of established sparsity properties of real-world images to the compressive sensing regime involving under-sampling with the linear operator \( \Phi \).
2.1.3 The $\ell_1$-MAGIC Library

Open source implementations of the algorithms which solve the convex optimization problems $\text{minDCT}$ and $\text{minTV}$ have been provided in the MATLAB computing language by the Candès and Romberg [32]. Both of $\text{minDCT}$ and $\text{minTV}$ fall into a class of optimization problems known as second order cone programs (SOCPs). In broad terms, the reconstruction algorithms iteratively improve an initial solution guess until some halting condition is achieved. The details of these algorithms as implemented by $\ell_1$-MAGIC, although beyond the scope and concern of this thesis, are described in detail in the library documentation, and can be found in Chapter 11 of the textbook by Boyd and Vandenberghe [33].

The optimization functions implemented by $\ell_1$-MAGIC naturally require additional step and tolerance parameters than those visible in the formal statements of $\text{minDCT}$ or $\text{minTV}$. These parameters are however non-physical in the context of reconstructing single pixel images. In all cases in this work, such additional parameters were assigned to the default values defined by the relevant $\ell_1$-MAGIC library functions.

2.2 The Role of Random Matrices

Of particular significance is the result that many classes of random measurement matrices are amenable to efficient CS reconstruction algorithms with high probability [34]. Suitable $M$ by $N$ measurement matrices $\Phi$ may be designed quite easily in practice by drawing the entries of $\Phi$ uniformly from a suitable probability distribution function. In this way, the entries of $\Phi$ may be considered as independent and identically distributed (i.i.d.) random variables. Some important examples include measurement matrices constructed by:

- i.i.d. sampling from the normal distribution with mean 0 and variance $\frac{1}{M}$,
• i.i.d. sampling from a symmetric Bernoulli distribution, e.g. $\Phi_{ij} = \pm \frac{1}{\sqrt{M}}$, and
• i.i.d. sampling from any subgaussian\(^3\) distribution.

Justifications of the above methods of measurement matrix design can be found in [35, 36].

In order to carry out compressed sensing, a signal sparsity matrix $\Psi$ must also be specified in addition to $\Phi$. It has also been shown [35] that such random measurement matrices are *universally applicable* to CS regardless of the sparsity matrix $\Psi$ of interest. Random measurement matrices therefore serve as universal measurement protocols for weighting the sampled linear signal value combinations.

### 2.3 Some Closing Remarks

A comprehensive survey of the compressive sensing literature would extend far beyond the scope of this thesis. A curated library of publications and learning resources maintained at \(<\text{http://dsp.rice.edu/cs}>\), for example, currently boasts *well over one-thousand* entries and will no doubt continue to grow.

For the purposes of this background chapter, a clear summary of the goal of compressive sensing and the rapid motivation of our reconstruction programs has taken precedence over a thorough description of the field. The main contributions of compressive sensing do not center around the relevance of assuming the compressibility of signals. What compressive sensing *does* contribute is a variety of tractable reconstruction programs, combined with theoretical guarantees of accuracy and performance. The nature of compressive sensing as a constellation of rigorous theorems should not be understated, despite their absence from the story told here.

---

\(^3\)Subgaussian distributions refer to any distributions whose tails decrease at least as fast as the tails of a normal distribution. More precisely, a random variable $X$ obeys a subgaussian distribution if for all $t > 0$, there exists positive constants $C$ and $k$ such that $P(|X| > t) \leq Ce^{-kt^2}$. 

We will later be in a position to comment on the performance of the two reconstruction programs, minDCT and minTV, in the context of the experimental portion of this thesis. It should be pointed out that neither reconstruction algorithm is by any means ‘state of the art’. The $\ell_1$-MAGIC implementations of minDCT and minTV, according to documentation [32], are also of a generic nature: “The code is not meant to be cutting-edge, rather it is a proof-of-concept showing that these recovery procedures are computationally tractable, even for large scale problems where the number of data points is in the millions”.

What has yet to be seen is the novelty of the compressive sensing framework to the applications of single pixel imaging. In this imaging regime, the optical properties of an object are sampled in an inherently linear fashion, with the underlying data often readily compressible. However, the potential of random sampling implemented with speckle patterns in single pixel imaging systems has yet to be fully realized. The understanding of this state of affairs is the topic of Chapter 3.
Chapter 3

Single Pixel Imaging

Imaging systems work by creating a one-to-one correspondence between conjugate points in an object plane and image plane. An image is the recording of spatial information regarding the structure of the light field at the imaging plane.

Despite the ubiquity of digital camera technology today, detectors for other ‘exotic’ wavelengths of light remain expensive outside of the sensitive range of silicon. In an effort to reduce the costs of imaging systems for non-visible light, a common tactic is the exchange of many detective elements for a single detective element with superior noise and sensitivity characteristics.

While the recording of light in the imaging plane is still possible simply by raster scanning the detector element across the imaging plane at the desired resolution, the total collection time is almost always limited by the speed at which the detector can be mechanically repositioned with high accuracy. Due to the overhead of repositioning, the collection of image data is often limited to a rate several orders of magnitude slower than the response time of the detector.

Single pixel imaging techniques improve upon raster scanning by eliminating the need for detector repositioning. This is done by modulating the light field at the imaging plane with a pattern of known spatial structure. Modulated light spanning the
entire imaging plane is subsequently directed onto the detector element, which records a single so-called bucket signal associated with the modulation pattern. The known modulation patterns and their associated bucket signals are then used to reconstruct the underlying light field at the imaging plane. In contrast to raster scanning, single pixel image acquisition speed is limited by the time required to modulate the light field at the imaging plane which can often be achieved at speeds approaching the response time of the detector.

The single pixel imaging technique differs from raster scanning in both the addition of a reconstruction step, and in the requirement of a method for generating a large number of known modulation patterns at the imaging plane. Fortunately, in many cases both of these requirements may be fulfilled separately from data collection. With a known library of modulation patterns available, a rapid sequence of bucket signals may be acquired from which a reconstructed image may be computed later.

The asymmetrical separation of the image sampling and reconstruction steps in a single pixel imaging experiment allows for improvements of the reconstruction method to take place independently from the context of the experimental setup. With the emergence of the mathematical theory of compressive sensing, exactly such an independent improvement has taken place. In order to better understand the advantages which compressive sensing affords single pixel imaging, this chapter has been divided into separate discussions of single pixel imaging experiments: those using compressive sensing-based reconstruction techniques, and those not.

### 3.1 Overview of Single Pixel Imaging

Single pixel imaging systems work by collecting bucket sums of a light field which has both interacted with an object under examination and a controllable modulator device. In practice, the order in which the imaging beam interacts with the ob-
Figure 3.1: An example of an active mode single pixel imaging system, defined by the arrangement [Modulator] \(\rightarrow\) [Object] \(\rightarrow\) [Bucket Sensor]. A lens initially collimates a point source, which illuminates the modulator and target pattern before being focused onto the bucket sensor at far right. The symbols \(\phi_i\), \(x\), and \(y_i\) correspond to the \(i^{\text{th}}\) modulator transmission profile, object transmission profile, and resulting bucket signal respectively. The transverse profile of the imaging beam along various sections is shown in dark gray.

ject and the modulator introduces an important distinction between two modes of operation unique to single pixel imaging systems. It will be useful to adopt the following terminology: the active mode defined by the arrangement [Modulator] \(\rightarrow\) [Object] \(\rightarrow\) [Bucket Sensor], and the passive mode defined by the arrangement [Object] \(\rightarrow\) [Modulator] \(\rightarrow\) [Bucket Sensor].

An example schematic of an idealized ‘active mode’ single pixel imaging system is shown in figure 3.1 which is setup to image a thin transparency of the letter ‘S’. Light from a collimated illumination source at left interacts with a modulation device which projects a pattern of structured light onto the target. After the target, a lens collects light and focuses it onto a bucket photodetector at right.

Let the vector \(x\) represent the discrete two-dimensional intensity transmission function of the imaging target across a defined set of \(N\) pixel locations, and let \(\phi_i\) \((i = 1, \ldots, M)\) represent the sequence of \(M\) light intensities produced by the modulator mapped onto the same pixel locations. We shall assume that the modulator operates by linearly modulating the amount of transmitted energy at each pixel location in the image plane. By definition, the result of the structured illumination of the imaging
target is the propagation of the intensity functions $y_i$ whose values at each pixel location are equal the product of the corresponding pixel values of $x$ and $\phi_i$. Due to the focusing action of the lens, the associated bucket signals $y_i$ are equal to the element-wise sum across the elements of $y_i$, and as a consequence we may express the bucket signals in terms of the inner product $y_i = \langle x, \phi_i \rangle$. By collecting the bucket signals into an $M$-dimensional column vector $\mathbf{y}$, the observed bucket signals and projected illumination patterns may be summarized in terms of a single matrix equation:

$$\mathbf{y} = \Phi \mathbf{x},$$  \hspace{1cm} (3.1)

where the $M$ by $N$ measurement matrix $\Phi$ is the matrix whose $i^{th}$ row is formed by the vector $\phi_i$. When viewed in terms of reconstructing the vector $\mathbf{x}$, equation (3.1) represents the condition that a solution reconstruction $\mathbf{x}$ be consistent with the observed data $\Phi$ and $\mathbf{y}$. However, it is only in restricted cases that the matrix equation (3.1) may be solved by direct inversion.

In the above example, the single pixel observations leading to the matrix equation (3.1) were conceived as a sequence of structured illuminations of the target transparency $x$. One may arrive at an identical formulation by instead exchanging the order of the object and modulator in the experiment, with or without the introduction of a focusing lens. Two canonical arrangements of a ‘passive mode’ single pixel imaging system are shown in figure 3.2. Placing the object before the modulator in a collimated beam path results in a functionally identical system to that shown in figure 3.1. By using a focusing lens, as in (b), a passive mode arrangement can function analogously to conventional photography by using only light emanating from an object scene to produce imagery. With the object illuminated by some external means, a converging lens is placed in the system such that the focal plane of the object’s real image is superimposed at the location of the modulator. A modulated real image is
CHAPTER 3. SINGLE PIXEL IMAGING

Figure 3.2: Two examples of *passive mode* single pixel imaging systems, defined by the arrangement [Object] $\rightarrow$ [Modulator] $\rightarrow$ [Bucket Sensor]. (a) A collimated beam illuminates the target whose shadow intensity pattern is given as input to the modulator. A collecting lens and bucket detector are at far right. (b) A point source back illuminates the target pattern $\mathbf{x}$ which is imaged by the lens to the real image intensity pattern $\mathbf{x}'$ at the modulator. Multiplexed light from the modulator is then collected by a lens onto the bucket sensor at far right. The symbols $\phi_i$, and $y_i$ correspond to the $i^{th}$ modulator transmission profile, and resulting bucket signal respectively. The transverse profile of the imaging beam after the lens is shown in dark gray.

then produced which is summed at the bucket detector.

The dual relevance of equation (3.1) in the alternate arrangement of figure 3.2 is clear by considering the real image intensity pattern $\mathbf{x}'$ formed of the target at the modulator. The output illumination pattern produced by the modulator under the constant input of the image structure $\mathbf{x}'$ is the sequence of structured propagations $\mathbf{y}_i$ summed at the bucket sensor, equal the product of the corresponding pixel values of $\mathbf{x}'$ and $\phi_i$. Consequently, a summary of the observations of the experiment shown in figure 3.2 can be stated in exactly the matrix form of equation (3.1) through the use of the inner product $y_i = \langle \mathbf{x}', \phi_i \rangle$. 
In conclusion, the ability to modulate intensity patterns at either the location of the object (active mode) or the object’s real image (passive mode) is the critical ingredient for single pixel imaging. Operating within this constraint, many variations of experimental setup are possible. For example, the use of collimated, non-diffracted beams in figures 3.1 and 3.2 is not necessary, nor is the restriction to 2D object patterns. Imaging systems can also be considered in which the modulator and/or object are placed in reflection to the imaging beam. It is even possible to impede the bucket detector with a scattering medium turbid enough to completely de-correlate the composite structure of the object/modulator intensity pattern.

3.1.1 Survey of Experiments

Disk Scanning Systems

The Nipkow disk technology behind the early development of television may be considered as the earliest example of a single pixel imaging system. By using a strong light source and a spiraled pattern of holes cut into a rapidly spinning disk, a ‘flying spot’ of illumination was scanned across a scene. An array of photodetector tubes placed nearby recorded the wide scattered light reflected from the object at the location of the spot. The amplified photocurrent signal could then be transmitted and used to modulate a light source placed behind a similar disk which was synchronized for real-time viewing. A period diagram of such a system is shown in figure 3.3. The Nipkow scanning disk may be considered as a special case of single pixel imaging of an object in reflection, using a modulator with only a single mechanically controlled transmission spot.

Nipkow scanning disks have been used in recent years as a replacement for raster scanning in various imaging systems due to their high spot positioning speed. For example, Nipkow disks are still used in modern confocal microscopy [38, 39]. More
Figure 3.3: An illustration of an early television system featuring a Nipkow scanning disk (public domain) [37].

recently, a single pixel THz imaging system based on a Nipkow disk was presented by Ma et al. [40]. A subsequent improved design for a THz Nipkow disk by Li et al. used surface etched Fresnel lenses to increase transmission through the disk pinholes [41].

**Structured Illumination Transforms**

Recent active mode single pixel imaging systems have demonstrated unique performance improvements over scanning disk systems by using modulation patterns with certain transform properties. By precisely controlling and judiciously selecting a set of structured illuminations, explicit solutions to the measurement matrix equation (3.1) can be obtained which offer various improvements in computing $\Phi^{-1}$, or in reducing the number of measurements necessary for successful imaging.

A passive mode THz single pixel imaging experiment by Shrekenhamer et al. demonstrated successful imaging using modulator patterns based on modified Hadamard matrices [42]. Spatial modulation of the THz beam was achieved using optically controlled photoexcitation in a semiconductor wafer which was structurally illuminated with a digital micromirror device (DMD). Each masking pattern was precisely con-
trolled in order to achieve a full-rank measurement matrix \((M = N)\) from which images were reconstructed by direct inversion of the measurement matrix (see equation (3.1)). By using Hadamard-based masking patterns to construct \(\Phi\), the matrix inversion step was reduced to a closed form solution which could be quickly calculated. Optically controlled THz modulation has also been used in a recent study by Stantchev et al. [43].

An active mode single pixel imaging system for thru tissue diagnosis has also been investigated by Durán et al. using modified Hadamard matrices projected using a DMD and white-light source [44, 45].

An active mode single pixel imaging system capable of directly obtaining the 2D Fourier transform coefficients of an imaging target through sinusoidal illumination patterns has been presented by Zhang et al. [46]. Modulation was achieved using a commercial light projector which displayed grayscale sinusoidal patterns of chosen spatial frequencies in the \(x\) and \(y\) directions. The real and imaginary parts of the corresponding transform coefficients were determined by differential bucket signal measurements recorded across four phase-shifted illuminations at each spatial frequency. High quality imaging was possible even in cases where the bucket detector was obscured with a glass diffuser, or illuminated only by a sheet of white paper which diffusely reflected light from the scene at a 90 cm distance. By illuminating the object scene in sequence starting with the lowest spatial frequencies and exploiting the sparse Fourier transforms of most natural images, recognizable reconstructions were possible after sampling only 1% of the spectrum coefficients.

Subsequent work by Liu et al. has demonstrated a full-color active mode single pixel imaging system which operates by obtaining the coefficients of the 2D discrete cosine transform (DCT) [47]. Structured illumination was achieved using a commercial light projector to modulate the sequence of grayscale cosine patterns. The
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A projector was used in tandem with a filtered photosensor which simultaneously measured bucket signals across three color channels. The orthogonal nature of the DCT coefficients [48] allowed for a fast inversion of the measurement matrix, as $\Phi^{-1} = \Phi^T$. Moreover, the sparsity of most natural images under the DCT allowed for good quality color image reconstruction from only 10% of the measurements found at the lowest spatial frequencies.

Computational Ghost Imaging

A distinct type of active mode single pixel imaging system using large numbers of highly randomized illumination patterns was first proposed by Shapiro [49] as a demonstration of the non-quantum mechanical nature of certain photon correlation experiments. Beginning in the mid 1990s, these experiments [50, 51] had demonstrated that an entangled pair of photons could be used for imaging in a setup where one photon was directed onto a target aperture and bucket sensor, and the other photon directed along a separate arm and detected by a separate incidence counter which was raster scanned across the beam path. The resulting photodetection statistics from a coincidence counter produced images of the target. The technique was termed ‘ghost imaging’ due to the fact that the spatially scanned photons which determined the image never passed through the target aperture [52]. Subsequent theoretical and experimental work however challenged the necessity of entangled photon statistics in interpreting these experiments. An experiment by Bennink et al. demonstrated that a similar imaging effect could be obtained with classically correlated imaging beams without resorting to two-photon statistics [53]. Another experiment presented by Valencia et al. replicated the previous imaging result without the use of entangled photons [52].

The effort to assert the classical nature of ghost image formation culminated in the proposal by Shapiro [49] who described a technique of computational ghost imag-
Experimental setup for computational ghost imaging. A computer controlled spatial light modulator shown at left determines the modulation patterns \( \phi_i(0) \) which are used to compute the propagated masks \( \phi_i(d) \) seen at the object. A collecting lens and bucket detector is shown at right (figure adapted from [54]).

Image formation by non-local quantum correlations was therefore precluded by the use of a single photosensor. A diagram of the setup for computational ghost imaging is shown in figure 3.4. A single frequency light source is used to illuminate a modulator which produces a deterministic intensity pattern far from the object. Next, the structured illumination pattern at the location of the object is obtained by numerically propagating the known patterns at the modulator. Transmitted light through the object is subsequently collected by a single bucket detector. Computational ghost imaging was later experimentally demonstrated by Bromberg et al. [54].

In practice, image results in computational ghost imaging are obtained as spatial correlations averaged over a large number of highly randomized bucket signals and modulation patterns. More precisely, images \( x \) are computed as the spatial cross correlation of the bucket signals \( y_i \) and computed masking pattern vectors \( \phi_i(d) \).
\[(i = 1, \ldots, M)\text{ in the limit of large measurements } M:\]

\[x = \langle \Delta y_i \Delta \phi_i(d) \rangle, \quad (3.2)\]

where \(\langle \cdot \rangle = \frac{1}{M} \sum_{i=1}^{M} (\cdot)\) denotes the ensemble average, and the bucket signal and masking pattern fluctuations are defined as \(\Delta y_i \equiv y_i - \langle y_i \rangle\) and \(\Delta \phi_i(d) \equiv \phi_i(d) - \langle \phi_i(d) \rangle\) respectively. The reconstruction method may be interpreted as follows. Masking patterns which happen to share a low correlation with the underlying transparency will produce a near zero bucket signal fluctuation \(\Delta y_i\). Masks sharing a high spatial correlation of their illumination pattern with the underlying object transparency will produce large positive fluctuations. Conversely, and most interestingly, masks sharing an opposite correlation and thus tending to illuminate only blocked object pixels will produce large negative bucket fluctuations. This implies that an \textit{inverted} illumination pattern will positively correlate with the underlying transparency. In this fashion, the image is summed as a weighted average of masking pattern fluctuation vectors, with non-correlating masks weighted low, strongly correlating masks weighted high, and inversely correlating masks weighted negatively to produce corrected patterns with a high weighting.

Conveniently, the spatial cross correlation may be computed as a running average. In practice however, the number of measurements needed to realize a stable image \(x\) via equation (3.2) can exceed the number of image pixels \(N\) by several orders of magnitude. In actual experiments [54], random phase masks have been used to provide the large number of masking patterns needed for computational ghost imaging.

As pointed out by Shapiro, in cases where the imaging wavelength leads to strong diffraction effects after interaction with the modulator, the resulting computed scalar field will vary strongly with the propagation distance towards the object. Therefore, in the context of correlation based reconstruction (3.2), the computational technique...
allows for a 3D sectioning of the object space light intensity from a single realized set of measurements $\{\langle \phi_i(0), y_i \rangle \}_{i=1}^M$. By varying the propagation distance used to compute the masking patterns, simultaneous reconstructions of an object and its diffraction pattern are possible [54].

3.2 Single Pixel Imaging with Compressive Sensing

The emergence of the theory of compressive sensing has led to a dramatic shift in the field of single pixel imaging. By assuming image sparsity with respect to a known basis or measure function, accurate solutions to the single pixel measurement equation (3.1) may be obtained even in cases where the number of measurements $M$ can be as low as 10% to 20% of the number of image pixels $N$, in contrast to the cases of disk scanning ($N$ measurements) and ghost imaging (far larger than $N$ measurements). We have however seen that a comparable reduction in the number of necessary image samples can alternately be obtained without the use of compressed sensing via a ‘lowest-spatial-frequencies-first’ ordering of structured illuminations determining the transform coefficients of an image. Remarkably, compressive sensing can reconstruct images in cases where the measurement patterns are randomly generated. As a result, no attention to the realized order of such masking patterns is necessary to achieve a reduction in measurements for images with underlying transform sparsity. The equal significance of all measurement pairs $\langle \phi_i, y_i \rangle$ also implies that image reconstruction is robust against the loss of recorded data. The particular transform which induces a sparse image representation need not even be known at the time of an experiment.
3.2.1 Survey of Experiments

The first compressive single pixel imaging system was demonstrated by Takhar and Duarte et al. based upon randomized passive mode sampling with a digital micromirror device (DMD) [1, 55]. A DMD is an addressable array of microscopic mirrors which are electrostatically actuated to selectively reflect portions of an incident beam in one of two fixed directions. A lens was used to image a small photograph target onto the DMD surface with one of the reflection arms directed onto a collecting lens and bucket detector. An LED source was used to illuminated the target in reflection. Color images could be obtained from separately reconstructed color channels by RGB filtering the illumination source. More recently, full color active mode single pixel imaging has been carried out by Welsh et al. using a high-speed commercial light projector and a dichroic beamsplitter for simultaneous measurements of the RGB color channels [56].

The first compressive THz single pixel imaging system was soon demonstrated in a series of experiments by Chan et al. who used random binary masking patterns to modulate a collimated beam from a THz time domain imaging system [57, 58]. The randomized masking was implemented using six-hundred copper printed circuit boards (PCBs) which were sequentially placed into the beam path along side a fixed transmission target. Each bucket signal corresponded to an entire coherently detected THz waveform which was sampled at the desired frequency. Remarkably, by virtue of the coherent detection system used, complex valued bucket signals could be extracted which resulted in the recovery of phase information at the imaging plane. Successful amplitude and phase imaging was later repeated in an experiment which exchanged the PCB masks for compressed sampling in the Fourier plane of a focusing lens [59].

Compressive sensing has also been used to improve the sampling efficiency of computational ghost imaging. In a direct extension of their earlier experimental work [54],
Katz, Bromberg, and Silberberg demonstrated ‘compressive ghost imaging’ by applying improved reconstruction techniques to their previous experimental data [60].

Compressive sensing techniques were used by Brady et al. to determine a 3D object from a Gabor hologram [61]. In general, a Gabor (on-axis) hologram is not sufficient to uniquely estimate a 3D object. Despite this, by approximating hologram formation using a linear model and interpreting a Gabor exposure as an undersampled representation of the 3D object, successful data-cube reconstructions were obtained. A later extension of this work enabled imaging in the THz region, using a 100 mW coherent source at 94 GHz [12].

A design for a passive millimeter wave imaging (PMMWI) system was first presented by Babacan et al. [62]. The proposed system featured a large single transmission mask with a cyclic row structure such that a relative one pixel translation of the mask under the imaging aperture yielded a new modulation pattern with sufficient differences [63]. Extensions of this work provided an explicit construction of the monolithic transmission mask, as well as an experimental actualization of the system [64, 65].

A THz transmission imaging system based on a monolithic spinning disk modulator has been demonstrated by Shen et al. [66, 67]. The Nipkow-inspired modulator disk was etched in copper on a PCB with random binary masking pixels. Sample transmission targets were placed near the disk perimeter which was illuminated with IR and THz-TDS beams parallel to the spinning axis. In operation, the disk could be spun continuously allowing for a much faster modulation speed than demonstrated in previous THz beam masking experiments [58, 57, 64, 65].

A compressive THz single pixel imaging system based on a metamaterial absorber has been demonstrated by Watts et al. [2, 3] which advanced the previous work by Shrekenhamer et al. [42]. Metamaterials are comprised of sub-wavelength conductive
structures which are capable of changing the dielectric properties of their support medium when a voltage is applied. By applying a bias to select ‘pixel’ elements in the metamaterial array, a real-time spatial modulation of the THz beam was achieved.

A comprehensive evaluation of THz compressive sensing for security screening applications has been presented in a doctoral thesis by Agustin [6]. A passive mode imaging system was first investigated which made use of a telescope arrangement and scanning focal spot actuated by a rotating parabolic mirror. Compressed sensing reconstruction techniques were shown to improve the trade-off between image quality and the spatial undersampling of the scanned focus spot. A single pixel imaging system was later constructed and evaluated which used an optically controlled semiconductor THz spatial light modulator [68]. A reduction in system cost was achieved by using an LCD transmission display, as opposed to a DMD, for modulator control.

Quite recently, Sun et al. demonstrated an active mode single pixel imaging system capable of estimating depth from a 3D scene using time of flight (TOF) measurements and compressive sensing [69]. By using a high speed photodetector and a pulsed laser source modulated with a DMD, specific scene depths could be targeted by reconstructing with bucket signals sampled from the photodetector current at the correct time offset. Compressive sensing techniques even allowed for depth mapped video output to be viewed in real-time.

3.3 Speckle Pattern based Single Pixel Imaging

I will now present the central ideas of this thesis in the context of THz single pixel imaging systems. The previous two sections of this chapter have provided a broad survey of the field of single pixel imaging across operating wavelengths in the visible and terahertz ranges. In particular for imaging at THz wavelengths, a large variety of spatial modulation techniques have emerged from the literature so far:
Printed Masking Patterns Several of the above cited works achieved THz beam modulation using printed circuit boards (PCBs) to control spatial transmission via chemical etching of the metal conductive layer. Although initial experiments relied on hundreds of individual mask sheets which had to be manually repositioned [57, 58], later mask constructions demonstrated fast imaging using monolithic printed patterns in the form of Toeplitz array constructions [63, 62, 64, 65] and rotating disks [66, 67].

Optically Controlled Semiconductors Some experiments achieved THz spatial modulation using optically controlled semiconductors [42, 68, 43]. With the use of a DMD or LCD transmission display, visible modulation patterns were projected onto a thin semiconductor disk causing a carrier-induced increase in THz absorption in regions illuminated with above band-gap optical excitation energies. Although this resulted in an increase in the cost and complexity of the system, the technique allowed for real-time modulation speeds with virtually no moving parts.

Metamaterial Arrays THz spatial modulation using metamaterials has been demonstrated in some recent single pixel imaging experiments [2, 3]. In contrast to optically controlled semiconductor modulation, THz metamaterial modulators function as non-mechanical stand-alone computer controlled devices with comparable refresh times to DMDs. However, these devices currently remain at the research cutting-edge, with demonstrated pixel resolutions still far below other methods. Despite these current limitations, continued advancement in the field of THz metamaterials can no doubt be anticipated.

All of the above methods suffer from a fundamental constraint related to the diffraction limit for THz radiation: the modulator device must be placed immediately adjacent to or in the real imaging plane of the scene or object. Modulator separation
from these positions leads to an immediate degradation in the measurements, as the forward propagated masking patterns may resemble random speckle patterns with little resemblance to the structure of the original masks.

To illustrate this point for the active mode, consider the previous situations depicted in figure 3.1 and figure 3.2 (a), which were shown with a non-diffraction limited collimated beam. However, in the diffraction limit, it is clear that successful imaging with $y_i$ and $\phi_i$ requires the imaging object and modulator to be placed adjacent to each other to minimize propagation effects. This parallels the constraints universally seen across the literature in corresponding experiments [58, 66, 67].

Conversely for the previous passive mode arrangement depicted in figure 3.2 (b), although the separation of the object and modulator was achieved with a lens, the accurate placement of the modulator at the real image position is mandatory to avoid the masking of a propagated real image. This constraint too is universally paralleled by the corresponding experiments in the literature [42, 43, 62, 64, 65, 2, 3].

The question is now asked: are other modulation schemes feasible for single pixel imaging systems in the diffraction limit?

As demonstrated by the principal of computational ghost imaging, image reconstruction is possible in an active mode single pixel experiment if the modulator input masks $\phi_i$ are computationally propagated forward to the target plane. It also stands in principal that such a propagation correction would enable successful reconstruction in a passive mode system in which a separation of the modulator from the real image plane of the target object was introduced. For such a computational method, it is mandatory to obtain the modulation intensity functions at some section of the imaging beam. For the variety of addressable ‘pixel-based’ modulators so far studied in THz single pixel imaging, this information is available from the control input to the
modulator. As the size of the modulator pixels approached the diffraction limit, such computed masking patterns would resemble highly random objective speckle patterns in the far field.

On the other hand, if a direct near field measurement of the modulated intensity patterns \( \phi \), at the object (or real image) plane were feasible, then any object producing a repeatable diffracted speckle pattern would be as equally suited for modulation purposes as any controllable pixel array within a computational scheme. With a means of obtaining direct measurements of the masking patterns, precise pixel control would be no longer necessary. At the price of carrying out *mask extraction*, the cost and complexity of current spatial light modulators could be exchanged for a much larger and robust class of devices.

The following modulation scheme for single pixel imaging is therefore proposed: that a sequence of masking patterns \( \phi \), be obtained by mechanically translating an optically rough reflective surface located at a distance from the object (or real image) plane. For such a system, imaging would occur in two separate, non-ordered steps: *mask extraction*, and *bucket collection*. Although in practice the necessary mask extraction step could be very time consuming, with sufficient translation repeatability the extraction step need only be carried out once during the life of the system. The stability of the modulator surface and the repeatability of the mechanical translations is therefore key for the efficient operation of the proposed system.

**Active Mode Imaging**

One possible arrangement for an active mode system is displayed in figure 3.5. In (a), the system is shown in a configuration for mask extraction. The modulator is a flat reflective surface which is mechanically repositioned in some repeatable fashion. With the source illuminating the modulator, a photodetector is raster scanned across the imaging plane where target objects are to be placed. One such raster scan is carried
Figure 3.5: An active mode imaging system using speckle pattern modulation. (a) Configuration for mask extraction by raster scanning. (b) Bucket collection, showing the insertion of an object target at the mask extraction plane.

out for each position of the modulator surface. The resolution and extent of the mask collection raster scans determines the number of pixels $N$ and image dimensions of the masks $\phi_i$ and reconstructed images. Owing to the advantages of compressive sensing based reconstruction, only a reduced number of masking patterns $M < N$ need be collected. In (b), the configuration for bucket collection is shown with an object placed at the imaging plane. A single bucket signal $y_i$ is then recorded as the modulator is successively translated into the same positions recorded during mask extraction. Following bucket collection, the experiment is complete. The recorded
dataset \( \{ (\phi_i, y_i) \}_{i=1}^M \) may then be used for any number of subsequent reconstructions.

Some details of the above example system are not requirements of the proposed technique. Firstly, the exact manner in which the modulator surface is translated is irrelevant, as is the modulator’s overall shape. Alternative methods include modulators implemented as a continuously spinning reflective disk, for example. The means used to characterize the masking patterns is also not important. With the correct compensations made during image processing, the mask extraction and bucket collection steps could be carried out using different photodetectors with completely different noise and responsivity characteristics.

The modulator translation and mask extraction scanning requirements of the above system depend upon an accurate mechanical means for positioning. Fortunately these requirements are easily met for operating wavelengths in the THz spectrum. In addition, the construction of a stable optically rough reflective surface for millimeter wavelengths can be carried out at minimal cost. As will be demonstrated in the next section, a successful single pixel imaging system was built which used a modulator constructed out of cardboard and conventional aluminum foil.

When used in an active mode imaging system, the proposed modulation scheme may be interpreted as the illumination of the target by a sequence of speckle patterns determined by the relative positioning of the modulator surface and radiation source. In summary comparison to the previous active mode THz experiments [58, 66, 67], the proposed system exchanges printed pseudo-random modulator patterns for a simple reflective surface with experimentally determined masking patterns allowing for object-modulator separation.

**Passive Mode Imaging**

The proposed speckle-pattern modulation scheme may also be employed as a replacement for printed masks in passive mode single pixel imaging with lenses. Two example
passive mode arrangements are shown in figure 3.6. In contrast to the active mode case of figure 3.5, the mask extraction step (a) involves the raster scanning of the radiation source to define the masking plane with the bucket detector stationary. Also, the use of a focusing lens with the bucket detector is not necessary. For each position of the source across the masking plane, a bucket detector signal is recorded as the corresponding pixel value of the masking pattern. In this way, a masking pattern \( \phi_i \) is obtained for each position of the modulator. Following mask extraction, two object insertion options are shown in (b) and (c). In (b), the object is simply inserted directly at the mask extraction plane. In (c), an object is positioned near a focusing lens so that its real image is focused at the mask extraction plane. Bucket collection then proceeds with the signals \( y_i \) recorded as the modulator is scanned across the mask positions. In either case, following bucket collection, the experiment is complete, with the recorded dataset \( \{ (\phi_i, y_i) \}_{i=1}^{M} \) being used to generate subsequent image reconstructions.

In contrast to the active mode example (figure 3.5), the raster scanning of the source during the mask extraction step in figure 3.6 resulted in a sequence of masking patterns \( \phi_i \) which did not correspond to any structured illumination or modulation at the image plane. Instead, the masking patterns may be interpreted as arrays of point-wise scalar transfer functions relating the overall transmission of radiation between the real image plane and the bucket detector due to the action of the modulator surface. During bucket collection, a real image formed at the object focal plane may be considered as a structured array of point sources at the scanned positions. For such a case, the total illumination seen at the bucket detector would equal the sum of the product of each image plane intensity with the corresponding transfer function scalar. The bucket signal could therefore be written as \( y_i = \langle x', \phi_i \rangle \) for a real image intensity pattern \( x' \) at the object focal plane. The recorded image measurements
Figure 3.6: Two passive mode imaging systems using speckle pattern modulation. (a) Configuration for mask extraction by raster scanning. (b) Target object back illuminated and inserted into the mask extraction plane. (c) Alternative arrangement, with the target object inserted in passive mode using a lens to place the focal plane at the plane of mask extraction.
\{(\phi_i, y_i)\}_{i=1}^{M} \) for the proposed passive mode system would therefore obey the single pixel matrix equation (3.1) and as a consequence, image reconstruction would follow from the identical methods for active mode imaging.

3.3.1 Discussion

Comparison to Ghost Imaging

A close comparison can be made of the proposed speckle pattern modulation scheme with computational and compressed ghost imaging. For the proposed active mode arrangement (figure 3.5), the main differences from computational ghost imaging are (1) the replacement of a computer controlled spatial light modulator for a simple optically rough reflective surface, and (2) the substitution of an experimental mask extraction step for the use of propagated intensity patterns computed from the SLM input masks. In addition, although the ghost imaging technique in principal could apply across any wavelengths, experiments so far have used visible laser light aligning with reliable methods for computer controlled modulation.

It is also clear that both techniques share identical capabilities allowing for the 3D sectioning of the object scattered light field. Although the mask extraction step of proposed method allows for the use of simple and inexpensive modulator devices, the placement of a framing aperture to limit modulator output could allow for a full field extraction of the masking patterns, and thus the use of computational propagation. Alternatively, 3D sectioning could also be carried out by obtaining multiple sets of extracted masking patterns at the desired field depths.

The solution for passive mode imaging within the proposed modulation scheme (figure 3.6) however does not have any known counterpart in the field of ghost imaging.
Reversal of Detectors and Sources during Mask Extraction

A result from the theory of scalar light diffraction approximated by the Fresnel-Kirchhoff formula is the symmetry with respect to the diffracted field produced under a reversal of the point locations of the source and observer, a property known as the Helmholtz reciprocity theorem [24]. More explicitly, the Fresnel-Kirchhoff formula implies that a diffracted field observed at a point \( P \) from a source positioned at a point \( P_o \) will be equivalent to that diffracted field produced at \( P_o \) from a point source at \( P \). Under the conditions of this diffraction theory, we can therefore suppose that the process of mask extraction depicted in figures 3.5 and 3.6 for the proposed active and passive mode experiments could in principal be carried out with the locations of the point-like radiation source and point detector reversed in the setup. This would allow for the point source or the point detector to be exclusively used for scanning during mask extraction for either modality.

Possibilities for THz Phase Retrieval

Phase retrieval from single pixel measurements has been demonstrated in several experiments using an augmented setup to extract bucket signal phase information. Single pixel phase reconstructions were first demonstrated by Chan et al. who used coherently detected THz waveforms to obtain complex bucket signal measurements with binary printed amplitude masks [59]. The proposed reflection modulator may in general represent a complex modulation of the light field, masking in both amplitude and phase. For the proposed method, it is anticipated that object phase retrieval is possible if the coherent detection of THz radiation is available for both the raster scanned mask extraction and bucket collection steps.

In cases where such coherent detection is not feasible, phase reconstruction can also be carried out from two intensity reconstructions taken of the object and its
CHAPTER 3. SINGLE PIXEL IMAGING

diffraction pattern using the Gershberg–Saxton phase retrieval algorithm [54, 70]. Using the proposed method, such reconstructions could be obtained simultaneously by using two sets of extracted masks obtained at the object (or real image) plane, and in the diffraction plane. Alternatively, such reconstructions could be obtained from a single set of extracted masks using computational 3D sectioning.

Application to Current Passive Imaging Systems

The proposed passive mode modulation scheme may have a relevant application to current PMMWI systems [64, 65]. Given the previously mentioned restrictions on modulator placement in current THz systems using physical masks, the replacement of a carefully designed and fabricated monolithic mask with a reflection modulator could result in a lower system cost and complexity. Moreover, the potential for separating the modulator from the system focal plane could lead to fewer restrictions on design. Finally, whereas monolithic modulator plates must be accurately indexed both horizontally and vertically, the increased mechanical speed of a spinning disk reflection modulator could impact the system acquisition time.
Part II

Experiment and Discussion
Chapter 4

Experimental Results

The imaging experiments presented in this thesis are the first demonstrations of speckle pattern modulation for single pixel imaging. Using the minDCT and minTV compressive sensing algorithms for reconstruction, transmission images of several binary target patterns were obtained in both the active and passive modes with masking patterns produced by an optically rough reflection modulator. For the active mode experiment, successful imaging results were consistently obtained with sampling rates as low as 30%, while in some cases recognizable images were obtained even with 10% sampling. For the passive mode experiment, stable imaging of a one-dimensional slot was possible by averaging several independent reconstructions at the level of 30% sampling.
Although the purpose of this chapter is to outline the methodology and results of the experimental work, the nature of the single pixel imaging technique entails a multitude of imaging results across reconstruction parameters in the context of any experiment. For the sake of brevity, only a limited number of imaging results were chosen to support the discussion in this chapter. A comprehensive gallery has therefore been provided in Appendix C (page 107) to which the reader is invited to skip ahead for an at-a-glance impression of the experimental results.

4.1 Apparatus and Imaging Setup

4.1.1 Emission and Detection of THz

Aside from various equipment needed for the automation of the imaging experiments, the apparatus depends only on two main pieces of specialized instrumentation: a coherent THz emitter and a suitable detector. Fortunately, both items are available today as mature stand-alone devices which are easily interfaced with computers.

THz Source

The coherent THz emitter was a model ‘S21E4’ sub-terahertz source manufactured by TeraSense Group, Inc. which operated at a continuous wave (cw) output of 102 GHz at 60 mW, corresponding to a wavelength of approximately 3 mm [71]. This device generates polarized radiation using a solid-state device known as a double drift region (DDR) impact avalanche transit time (IMPATT) diode. When held under a reverse bias near the breakdown voltage, these diodes are capable of maintaining single frequency oscillations which are determined by the transit time of charge carriers moving through a doped transit region [72, 73]. In operation, the injection of charge carriers (electrons + holes for DDR diodes) arises from an ‘avalanche’ chain of impact ioniza-
Figure 4.1: Photographs of the THz source (a) and imaging camera (b).

ions triggered by the applied field briefly crossing the threshold to cause breakdown. In this way an applied AC voltage (in addition to the bias voltage) can modulate the current through the diode. A phase change between the applied AC voltage and the external circuit current arises naturally due to the time delay required for the avalanche buildup of charge carriers and their passage through the transit region. By carefully choosing the width of the transit region, a 180° phase shift between the AC voltage and external current can be achieved for a desired frequency. The result is a configuration for stable oscillation at THz frequencies.

Radiation is directed out of the front of the device through a conical metal tube approximately 3.4 cm in length. The tube has an internally machined taper which faces inwards from a circular opening 1.2 cm in diameter. In figure 4.1 (a), a white plastic end cap is shown covering the open end of the tube. The plastic end cap, used to blocks debris from entering the unit, is mostly transparent at THz frequencies and may be left in place when the THz source is in use.

The THz source is operated on an ON/OFF basis controlled by a physical switch. Although the source is capable of external switching and modulation, these features were not used in this study. The device is powered directly from AC mains using a
wall adapter (24 V, 18 W) supplied with the unit. The lifetime and performance of IMPATT diodes can depend strongly on their operating temperature. As a precaution against overheating, the external case of the IMPATT source was air cooled with an electric fan during extended use.

**Imaging Camera**

The detection of THz radiation was accomplished using a model ‘T15/16/16’ sub-terahertz imaging camera also manufactured by TeraSense with various sensitivity bands in the range of 50 GHz to 700 GHz at room temperature [74]. The device consists of a 16x16 (256 element) array of polarization sensitive THz detectors combined with a module for real-time data transfer over a USB interface. The dimensions of each pixel are approximately 1.5 mm square with the square aperture window of the camera measuring 3 cm by 3 cm. Using the viewing software provided, the camera produces a video-like stream of image captures with an adjustable frame rate.

The image data produced by the T15 camera is normally processed after acquisition. As a result, numerical values returned by each pixel are automatically mapped to the interval $[0, 1]$. The relative amplification of the sensed radiation is determined by the camera’s *exposure setting* which determines the time taken to integrate the value of each pixel in the sensor array. Each frame may also be computed as the accumulated average of a specified number of sequential measurements at a given exposure. Although this *accumulation setting* does not affect the camera’s operating frame rate, the exposure and accumulation settings together affect the camera’s noise characteristics. Operating at minimal exposure, the camera is capable of acquiring frames at a rate of 50 Hz.

The T15 camera may be controlled programmatically through a software library (API) provided by TeraSense. Imaging data from the camera is collected in an on-demand fashion via function calls to the API, which also allows for control over the
camera exposure and accumulation settings.

Although in principal the operation of a single detector element is all that is required for single pixel imaging, the use of a multi-pixel detector greatly simplified the experimental setup, allowing for the fast collection of masking patterns as well as eliminating the need for focusing optics. In practice the T15 camera was utilized as a ‘bucket detector’ by operating on a direct element-wise sum across all 256 pixel elements.

### 4.1.2 Reflection Modulator

![Figure 4.2: Closeup of the modulator surface shown at actual size. The divisions of the scale bar show distances of 1 cm.](image)

The reflection modulator, owing to the macroscopic wavelength of the coherent source, was compiled from common household materials found at minimal cost. The reflection modulator was constructed using aluminum foil adhered to a stiff construction paper backing with a spray adhesive. Before adhesion to the backing, the aluminum foil was manipulated by hand until a crumpled texture was achieved which varied spatially on a millimeter length scale. A closeup of the modulator surface texture is shown in figure 4.2. During experiments, the modulator was securely fastened against a wooden panel which was mounted vertically. The textured surface of the modulator measured 23 cm by 38 cm.
4.1.3 Automation of the Experiment

A fully automated imaging setup was implemented for the collection of experimental data. The setup involved programmatic control of both the imaging camera and the incremental positioning of the modulator and THz source.

Motorized Positioning Stages

The accurate positioning of equipment was achieved with the use of several LTS300 motorized positioning stages manufactured by THORLabs Inc. [75]. These stages can be manually operated or computer controlled through a USB interface. Programmatic control is supported through an API provided by THORLabs. The stages each have a 30 cm range of travel, with a positional repeatability of less than $\pm 2 \, \mu m$ [75].

Automation Software

Computer control of both the T15 camera and LTS300 stages was implemented within the MATLAB scripting environment using the respective API provided for each device. The automation scripts for data collection were executed on a desktop computer which was connected to the camera and positioning stages via USB.

4.1.4 Reconstruction Algorithms

The two reconstruction algorithms described in Chapter 2 (minDCT, minTV) are used for comparative image reconstructions in this section. Each algorithm, in addition to the masking pattern and bucket signal pair data form the experiment, must accept a single accuracy parameter $\epsilon > 0$ which strongly determines the reconstruction quality. Unfortunately, $\epsilon$ parameter values leading to accurate reconstructions depend on the sampling number $M$, image pixel count $N$, and the nature of the underlying image $x$. An iterative method for estimating appropriate values of $\epsilon$ has therefore been derived
in Appendix B. The result is a reduction in the likely space of possible values of \( \epsilon \) from any positive real number, to a tuning factor in the range of \([0, 1]\) in all cases of \( M, N, \) and \( x \).

Each image reconstruction presented below and in the reconstruction gallery Appendix C has an associated tuning factor value determining the parameter \( \epsilon \). In the course of generating reconstruction results, the tuning factor was adjusted by hand until a ‘best quality’ image was obtained. In all cases the tuning factor value fell within the expected \([0, 1]\) range.
4.2 Active Imaging

An illustration of the setup used for active mode single pixel imaging is shown in figure 4.3. A corresponding photograph of the same setup is shown in figure 4.4.

To begin the collection of an image, a set of modulator positions was first recorded based on the indexed positioning of the motorized translation stage. The modulator was then illuminated by the THz source creating a speckle pattern seen at the camera. Masking patterns were then recorded for each modulator position directly as 16 by 16 pixel frames obtained from the camera within the setup geometry. The camera’s multi-pixel sensor array therefore determined the resolution and extent of the imaging plane in this experiment.

Following mask collection, the camera was repositioned to allow for imaging targets to be placed at the imaging plane of the collected masking patterns. This was accomplished by means of a small backwards horizontal offset of 0.150 inches (3.81 mm) of the camera away from the modulator. This backwards offset was chosen so that an imaging target adhered flat against the camera’s front enclosure surface would occupy the mask collection imaging plane following repositioning. The backwards offset was initially determined experimentally by varying the offset distance while observing reconstruction quality. Later, following an internal examination of the imaging camera, the backwards offset distance was confirmed as the approximate optical path length between the front surface of the camera enclosure and the internal sensor array in a direction normal to the camera aperture.

After repositioning of the camera, imaging targets were secured directly to the front face of the camera enclosure using an adhesive aluminum foil tape. Bucket signals were then recorded as the modulator was indexed to a subset of the positions chosen for mask collection. Each bucket signal was computed from a 16 by 16 pixel camera frame as the total element-wise sum across all pixel values.
CHAPTER 4. EXPERIMENTAL RESULTS

Figure 4.3: Several perspective drawings of the active mode imaging configuration used for experiment. (a) An elevated view of the setup, with arrows added to illustrate the horizontal axis along which the modulator was positioned. (b) Another slightly elevated view of the setup. The small square sensor aperture of the camera is visible near center. (c) A labeled, top-down view of the setup with distances added.
Imaging was then completed using the reconstruction algorithms (minDCT and minTV) which made use of solely the captured masking patterns and pixel-summed bucket signals. Following mask collection, and owing to the high precision of the motorized stages, the imaging of subsequent targets could proceed indefinitely, requiring only the collection of new bucket signals.

4.2.1 Experimental Results

Following the above procedure, single pixel transmission images were obtained of the seven binary imaging targets shown in figure 4.5. Each target was constructed out of adhesive aluminum tape applied to a paper backing and then laminated between two layers of adhesive cellophane tape. For the calibration purposes of the noise model (see Appendix B), images were also taken of the camera aperture in the completely blocked...
CHAPTER 4. EXPERIMENTAL RESULTS

Figure 4.5: The seven binary imaging targets used for experiment. (a) Photograph of the imaging targets. The imaged area of each target measured 2.5 cm by 2.5 cm and was closely sized to match the aperture window of the camera. (b) Corresponding direct transmission images of the imaging targets. Each transmission image was directly exposed at a distance of 37 cm from the THz source, and with the target adhered flat against the camera’s front enclosure surface in the same position as used in experiment.

and completely open states, leading to a total of nine imaging target configurations.

The imaging plane was defined by the 16 by 16 pixel \((N = 256)\) resolution of the camera. For mask collection, a complete sequence of 256 masking patterns was obtained with the camera’s highest averaging setting (100 accumulated frames). A sample of the collected masking patterns is shown in figure 4.6. The corresponding recorded modulator positions were equally distributed across 28 cm of the 30 cm extent of the motorized stage, with the distance between adjacent modulator positions equal to approximately 1.1 mm. The duration of mask collection was 49 minutes 30 seconds.

For each of the nine target configurations, a complete sequence of 256 bucket signals was recorded with reduced averaging (5 accumulated frames). The duration of point collection was approximately 7 minutes 20 seconds for each target pattern.
Figure 4.6: A sample of the masking patterns collected in the course of the active mode imaging experiment. The masks were collected consecutively between translations of the modulator. Across the sixteen masks shown, the total translation of the modulator was 1.7 cm.
Reconstructions were then carried out for various compressed sampling rates with data drawn from the complete collection of 256 measurements. For each sampling rate, a random subset of the full measurement collection was provided to the algorithms. As a consequence of selecting measurements at random, each imaging target presented a very large number of possible reconstructions within the full measurement set. Images averaged across several independent reconstructions were therefore also produced to provide a more stabilized record of reconstruction quality.

A set of reconstructions of target $\mathbf{H}$ (see figure 4.5) is shown in figure 4.7 to illustrate the variability in imaging results at the 30% sampling level without averaging. For each reconstruction, a random subset of measurements of size ($M = 77$) was selected from the complete collection of 256 measurements. Each reconstruction is recognizable to $\mathbf{H}$ with variable distortion. Results averaged across 10 reconstructions of $\mathbf{H}$ at 30% sampling are shown for comparison in figure 4.8 along with corresponding non-averaged results.

The expected range $[0, 1]$ of the imaging pixel values was not assumed in either of the reconstruction algorithms. As an indication of the success of the reconstructions to fit this range, a re-coloring of the $\mathbf{H}$ target imaging results is also shown in the right half of figure 4.8, with the colors blue and red corresponding to pixel values less than 0 and greater than 1, respectively.

Finally, a complete set of imaging results for all seven of the imaging targets (see figure 4.5) using both minDCT and minTV reconstructions is shown in figure 4.9 for 10 averages at 30% sampling.
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Figure 4.7: A sample of reconstructions of the target \( \mathbf{H} \) taken at a sampling rate of 30\% \( (M = 77) \). The top and bottom rows show reconstructions computed using the minDCT and minTV algorithms respectively. The value of the tuning factor used across all reconstructions was 0.25.

Figure 4.8: Gray scale and overshoot-colored reconstructions of the target \( \mathbf{H} \) taken at a sampling rate of 30\% \( (M = 77) \). (Left side) Sample reconstructions computed without averaging \((a), (b)\), and with 10 averages \((c), (d)\). The minDCT algorithm was used to compute images \((a)\) and \((c)\), while images \((b)\) and \((d)\) were computed using the minTV algorithm. (Right side) Corresponding re-colored versions of the images \((a)-(d)\) with the colors red and blue highlighting pixel values above and below the physical range of \([0, 1]\), respectively. The value of the tuning factor used across all reconstructions was 0.25.
Figure 4.9: Collection of imaging results for all seven target masks shown in figure 4.5 computed using the (a) minDCT and (b) minTV algorithms. Each result was averaged across 10 reconstructions sampled at 30%. The tuning factor values used for each reconstruction are shown below their respective images and were kept the same for each target.

As a quality metric for the reconstructed images, we may compare the reconstructed images $\mathbf{x}$ to reference ‘ideal’ images $\tilde{\mathbf{x}}$ by evaluating the mean squared error (MSE), defined as

$$\text{MSE} \equiv \frac{1}{N} \sum_{i=1}^{N} (x_i - \tilde{x}_i)^2.$$  \hspace{1cm} (4.1)

Good quality reconstructions should therefore exhibit small values for the MSE, in particular when compared to the maximum expected value for pixels in the image. We may evaluate the relative sizes of the MSE and peak image value using the peak signal to noise ratio (PSNR) [64], defined as

$$\text{PSNR} \equiv 10 \log_{10} \left( \frac{V_{\text{max}}^2}{\text{MSE}} \right),$$  \hspace{1cm} (4.2)

where $V_{\text{max}}$ represents the maximum value allowed for image pixels. In our case for
physical transmission/reflectance function values, we set $V_{\text{max}} = 1$. As can be seen from equation (4.2), the PSNR takes on larger values with increasing reconstruction quality. For comparison, a PSNR of 0 corresponds to the equal magnitude of the MSE and peak image value, with a PSNR of 10 corresponding to the case $\text{MSE} = V_{\text{max}}/10$.

Comparison error bar plots of the PSNR of the reconstructed \textbf{H} target are shown in figure 4.10 for reconstructions with and without averaging. As expected, the figures indicate that higher sampling ratios generally correspond to improved reconstruction quality. Moreover, averaging lead to an improvement in image quality with improved consistency. At 90\% sampling, the minDCT algorithm was found to occasionally produce anomalous low quality reconstructions. Such anomalous reconstructions have been indicated as outliers in the figure, chosen based on having a distance greater than 2.5 standard deviations from the mean.

The imaging results shown above are only a small impression of the many possible reconstructions when sampling rates and averaging are varied. For convenience, a more comprehensive gallery of the reconstruction results for all nine target configurations (including those for the completely open and closed aperture) is presented in appendix C.1 for a range of sampling rates both with and without averaging.
Figure 4.10: Peak signal to noise ratio (PSNR) vs. sampling ratio for reconstructions of the mask, compared against an ideal rendering shown at left. The error bars show ±1 standard deviation computed from 100 reconstructions at each point. (a) Reconstructions with no averaging. Five outliers beyond 2.5 standard deviations are shown as black dots for the 90% minDCT data point. The plots have been given a slight horizontal offset for clarity. (b) Reconstructions with 10 averages. Two outliers beyond 2.5 standard deviations are shown as black dots for the 90% minDCT data point. The value of the tuning factor across all reconstructions was 0.40.
4.3 Passive Imaging

Passive mode single pixel imaging was carried out for a one dimensional ‘line scan’ imaging plane defined by a slot shaped aperture. A perspective drawing of the setup is shown in figure 4.11, with a corresponding photograph shown in figure 4.12.

The slot aperture was constructed similar to the imaging targets used in the active mode experiments, as a layer of adhesive foil adhered to paper and laminated between two layers of transparent adhesive cellophane tape. In order to improve dimensional stability, the cellophane lamination was applied to completely span the aperture opening. During the experiment, the slot aperture was securely positioned and held taut between metal rails which were fastened by bolts.

Mask collection was carried out by scanning the output tube of the THz source across the slot aperture for each position of the modulator. This involved placing the opening of the THz source directly flush to the aperture surface with the end cap removed (shown in figure 4.11 (a) and (b)). Through the aperture slot, the source illuminated the modulator surface producing a speckle pattern seen at the camera. The resolution and extent of the source scan and slot aperture therefore defined the imaging plane for the line scans. For each position of the THz source across the aperture slot, a pixel-summed bucket signal was recorded on the camera. The array of bucket values taken for each position of the source across the aperture slot defined the masking pattern for each modulator position.

Following mask collection, the THz source was moved backwards into a position in which it illuminated the entire aperture slot. This configuration is illustrated in figure 4.11 (c) and figure 4.12. The camera and slot aperture were not moved. Imaging targets were then placed directly at the slot aperture. Bucket signal collection proceeded with the collection of pixel-sum bucket signals captured at the camera as the modulator was repositioned over a subset of the mask positions with the THz
Figure 4.11: Several perspective drawings of the passive imaging configuration used for experiment. (a) A slightly elevated view of the setup, with arrows added to illustrate the positioning axes indexed by the motorized positioning stages. The THz source is shown positioned immediately behind the slot aperture in the position for mask extraction. (b) Another elevated perspective of the setup in the mode for mask extraction. (c) A labeled, top-down view of the setup with distances added, with the THz source shown in position for bucket collection. Additional foil shielding (not shown) was used to prevent direct camera exposure from the THz source during mask extraction and to further block radiation around the edges of the slot aperture.
As with the active mode experiment, imaging was completed using the reconstruction algorithms ($\text{minDCT}$ and $\text{minTV}$). Following mask collection, the imaging of subsequent targets could proceed immediately with the collection of bucket signals.

4.3.1 Experimental Results

Following the above procedure, one-dimensional passive mode single pixel images were recorded along the opening of the slot shaped aperture. In practice, foil adhesive tape was used to block portions of the aperture window to create the target patterns for imaging. Two such target patterns were imaged in addition to the noise model.
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Figure 4.13: An illustration of the slot aperture and the two target patterns used for imaging. The slot aperture (actual size) is shown at the top superimposed with the twenty positions scanned by the THz source during mask collection. At bottom, the target patterns (i) and (ii) are shown along with associated plots of the expected transmission functions across the twenty discrete pixels of the line scan.

calibration images (aperture open and completely blocked) for a total of four passive imaging arrangements. The two non-calibration target patterns, along with an actual size illustration of the slot aperture, are shown in figure 4.13.

The one dimensional imaging plane was defined by the extent of the slot aperture divided equally into twenty positions of the THz source, as illustrated in figure 4.13. The resolution of the line scan images was therefore 1 by 20 pixels, with the distance between adjacent pixels equal to approximately 5 mm.

During the mask collection step, a full set of 20 masking patterns was obtained across a 26 cm translation distance of the modulator. Each of the 20 masking patterns required the THz source to be fully scanned across the 20 pixels of the aperture opening for a total of 400 camera measurements. A sample of the collected masking patterns is shown in figure 4.14. In order to reduce the collection time for these measurements, the accumulation function of the camera was not used. The total duration of mask collection was 12 minutes.
Figure 4.14: A sample of the masking patterns collected in the course of the passive mode imaging experiment. The masks were collected consecutively (top to bottom) between horizontal translations of the modulator. Across the eight masks shown, the modulator was translated a total of 9.6 cm.

For bucket collection, full 16 by 16 pixel captures from the camera were summed for the point measurements. For the collection of the bucket signals, the accumulation function of the camera was not used.

Image reconstructions were then carried out in the same way as for the active imaging results using the minDCT and minTV algorithms. For various compressed sampling rates, data was drawn randomly from the complete collection of 20 measurements. As a consequence of selecting measurements at random, each imaging target presented a very large number of possible reconstructions which varied widely. Therefore, images averaged across several independent reconstructions were therefore also produced to provide a more stabilized record of reconstruction quality.

A set of reconstructions of target (i) (see figure 4.13) is shown in figure 4.15 to illustrate the variability in imaging results at the 30% sampling level without averaging. For each reconstruction, a random subset of measurements of size ($M = 6$) was
Figure 4.15: A sample of reconstructions of target (i) taken at a sampling rate of 30% ($M = 6$) without averaging, showing a high degree of variability between reconstructions. The top and bottom rows show reconstructions computed using the minDCT and minTV algorithms respectively, with the expected transmission function superimposed in gray. The value of the tuning factor used across all reconstructions was 0.38.

selected from the complete collection of 20 measurements. As the figure demonstrates, the variability in the line scan reconstruction results is large enough to obscure the underlying pattern in most cases.

The impact of increased averaging on the line scans is illustrated in figures 4.16 and 4.17, which show 30% sampled images respectively of the targets (i) and (ii) compared at 10 and 50 averages.

Comparison error bar plots of the PSNR of the reconstructions of target (ii) are shown in figure 4.18 for reconstructions with and without averaging. For the minDCT algorithm, an expected increase in reconstruction quality with sampling ratio is observed. Interestingly, the opposite is observed for the minTV algorithm which for the line scans generally recovered the locations of target edges but not the value of contiguously dark regions. However as the figure shows, the PSNR performance of both algorithms is improved with averaging.
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Figure 4.16: Reconstructions of target (i) taken at a sampling rate of 30% ($M = 6$) with varying degrees of averaging. The top and bottom rows show reconstructions computed using the minDCT and minTV algorithms respectively, with the expected transmission function superimposed in gray. No averaging was used for the left column images (a) and (d), 10 averages were used for the middle column images (b) and (e), and 50 averages were used for the right column images (c) and (f). The value of the tuning factor used across all reconstructions was 0.38.

Figure 4.17: Reconstructions of target (ii) taken at a sampling rate of 30% ($M = 6$) with varying degrees of averaging. The top and bottom rows show reconstructions computed using the minDCT and minTV algorithms respectively, with the expected transmission function superimposed in gray. No averaging was used for the left column images (a) and (d), 10 averages were used for the middle column images (b) and (e), and 50 averages were used for the right column images (c) and (f). The value of the tuning factor used across all reconstructions was 0.42.
Figure 4.18: Peak signal to noise ratio (PSNR) vs. sampling ratio for reconstructions of target (ii), compared against the 20 pixel ideal reconstruction shown at left. The error bars show ±1 standard deviation computed from 100 reconstructions at each point. (a) reconstructions with no averaging. (b) reconstructions with 50 averages. The plots have been given a slight horizontal offset for clarity. The value of the tuning factor across all reconstructions was 0.50.

Just as for the case of active mode imaging, the imaging results shown in this chapter are only a small impression of the many possible reconstructions when sampling rates and averaging are varied. For convenience, a more comprehensive gallery of the reconstruction results for all four target configurations (including those for the completely open and closed aperture) is presented in appendix C.2 for a range of sampling rates and degrees of averaging.
Chapter 5

Summary, Discussion, and Future Work

5.1 Discussion of Experimental Results

The results collected in both Chapter 4 and the gallery Appendix C demonstrate the consistent recovery of THz single pixel images at sampling rates of 30% or lower. Between the two distinct compressive sensing reconstruction algorithms which were used to compute each image (minDCT and minTV), all results showed comparable image quality and behaviour with decreasing sampling. By averaging across 10 to 50 separate reconstruction results, images were obtained which stabilized into recognizable depictions of each target masking pattern.

For the active mode experiment, recognizable reconstructions were consistently obtained at the 20% level following 10 reconstructed averages. Qualitative differences in the reconstruction results obtained with the minDCT and minTV algorithms were also observed. Spurious content in the minDCT reconstructed images resembled periodic content at high spatial frequencies which resembled discrete cosine basis functions. For the minTV reconstructed images, high frequency content was rarely observed as
expected, with spurious image content resembling contiguous light colored regions in the image background.

For the passive mode experiment, image reconstructions generally featured a higher level of variability between reconstructions sampled at the same level. Whereas with the active mode experiments satisfactory image stability was seen with only 10 averages, some passive mode results required 50 averages to achieve stable recoveries. In addition, for the two partially blocked passive mode images, average-stabilized reconstructions did not converge to the underlying transmission functions everywhere. Although the cause of these inaccuracies is not presently known, likely contributors were the relatively small pixel resolution \((N = 20)\) and the method of mask extraction (discussed in the following section). Importantly however, the averaged-stabilized passive mode reconstructions did accurately reproduce the location and direction of the opacity changes of the masking patterns.

Across both of the active and passive mode experiments, the completely open and blocked calibration images demonstrated extremely consistent flat-image reconstructions even at sampling rates as low as 10%. The robust quality of these reconstructions at very low sampling rates is however perfectly consistent with the fact that such flat-images are examples of minimum sparsity signals according to both the discrete cosine basis, and the total variation.

Almost all image targets across both experiments demonstrated improved image contrast and quality with increasing bucket sampling. Interestingly, in a few cases, increasing the sampling rate from 50% to 90% resulted in a decrease in reconstruction contrast. Anomalous decreases in image reconstruction quality were however observed for near-complete measurement sets reconstructed at rates approaching 100% sampling. This is however likely due to the implementation of the optimization routines of the \(\ell_1\)-MAGIC library which in some ways depend on the assumption that \(M \ll N\).
Figure 5.1: Comparison images (16 by 16 pixels) illustrating the variable responsivity of the sensor elements of the THz camera. (a) Normally recorded image. (b) Image recorded by scanning the pixel in the 4th row and 4th column (from the top left of the pixel array) across the same pixel locations sampled in (a).

for the measurement collection size $M$ and image pixel count $N$. Some additional misbehavior of the $\ell_1$-MAGIC routines was also consistently observed in the sampling edge cases $M = N$, $M = N - 1$, and $M = N - 2$.

\subsection{5.1.1 Improvements to the Experiment}

Mask Extraction

In the active mode imaging experiment, masking patterns were collected as 16 by 16 pixel images recorded with the use of a THz imaging camera. The use of a multi-pixel camera, although not strictly necessary, greatly increased the speed of the mask collection process. However, variations in the responsivity of pixel elements were noted across the sensor plane. A comparison of a speckle pattern recorded normally using the camera and obtained by scanning a single detector element are shown in figure 5.1. The comparison shows visible differences between the normal and scanned images, with the scanned image featuring an enhanced smoothness. Although the software provided with the camera had the ability to implement per-pixel corrections for flat field normalization, a factory-set calibration file shipped with the camera was used to reference normalization across all camera measurements.
In the passive mode experiment, masks were collected by directly scanning the THz source’s circular opening across a slot shaped aperture (figure 4.13). The circular shape of the output opening however meant that the chosen positions of the THz source along the slot were chosen based on a balance between maximizing aperture coverage and minimizing pixel overlap. In designing the mask extraction procedure for the passive mode experiment, placing a square shaped aperture over the source opening as a means to define the pixel extent was considered, but not used based on the potential for disruptive feedback radiation reflected towards the IMPATT diode. A potential improvement for this mask extraction procedure would be to define the masking pattern pixel extents using a square aperture scanned across the slot and illuminated from the source in the position for bucket collection (figure 4.11). Such an aperture defining the pixel extents could also be implemented during mask extraction with a scanning photosensor.

**Reflection Modulator**

Only minimal design criteria were considered in the construction of the reflection modulator. In particular, the randomly varying sub-wavelength surface texture was approximated by eye following repeated manipulations of the foil by hand. Mechanical stability of the modulator was then achieved by adhering and partially flattening the foil matrix to a construction paper backing which was then firmly attached to a wooden backing board mounted on a motorized translation stage. The granular diffraction hallmarks of a speckle pattern were then confirmed via far field viewing with the THz camera. Translations of the camera in directions normal to the modulator surface also demonstrated a rapid spatial variation and grain structure which increased in size at large distances from the modulator.

The single pixel operation of the speckle pattern modulator depends on having repeatable, bright speckle patterns with a high variability and with grain sizes on
the order of the smallest features of the object. Various ad hoc improvements to the modulator system can therefore be suggested. According to the equation (1.1) for speckle grain size, smaller speckle sizes can be achieved by decreasing the separation from the imaging plane, or by increasing the illuminated modulator area. It is stressed that the in-plane translation of a flat modulator surface is only one possibility for the production of far field speckle patterns. Other possibilities include the addition of secondary optically rough reflectors or the addition of lenses or other optical elements. Such additional elements would not necessarily require mechanical positioning. A curved yet optically rough modulator surface could also be used to partially focus the speckle field in order to increase brightness.

5.2 Diffraction Limited Imaging

Due to diffraction effects, imaging systems which collect light in the far field cannot resolve distances smaller than the wavelength of the probing radiation. Single pixel imaging experiments which place the bucket photosensor in the far field from the object are therefore diffraction limited. In response to the experimental use of wavelength-sized pixels in THz spatial light modulators, the first investigation of diffraction limit effects on conventional masked single pixel imaging has very recently been presented by Schmitt and Rahm using a 2D simulation with modulated boundary conditions at THz frequencies [76]. Although the proposed active modulation scheme involves masking patterns realized as near field objective speckle pattern images, speckle pattern single pixel imaging is diffraction limited by the object scattering of modulator light and its bucket collection in the far field. In the passive mode, near field speckle patterns representing imaging plane to detector intensity transfer functions again give way to the limits of far field bucket collection as highly diffracted light at the object is widely scattered.
An additional diffraction effect is observed for single pixel imaging experiments. In the limit of small aperture size $d$, a $d^3$ relationship is predicted for the energy coupled through an aperture in an infinitely thin metallic sheet, with additional non-linearity introduced by considerations for finite aperture thickness [23]. In such cases the assumed linear relationship of the single pixel matrix equation (3.1) breaks down, along with the techniques of linear algebra at the heart of the compressive sensing. In their recent study, Shmitt and Rahm used direct matrix inversion to solve the matrix equation. To the present author’s knowledge, no studies are yet available regarding the effects of diffraction non-linearities on single pixel image reconstruction with compressed sensing.

5.3 Future Work

Immediate extensions to the experiments presented in this thesis would include imaging targets in the reflection geometry, and the use of targets with more general transmission functions. In addition, the use of a multi-pixel THz camera for convenient mask extraction and bucket collection was non-essential, nor was the use of 102 GHz radiation in the ‘low end’ THz region.

An important extension of the modulation scheme to pulsed THz time-domain systems could also provide multi-spectral single pixel imaging with the possibility of phase retrieval. Alternatively, the retrieval of phase information from single pixel images could also be investigated with cheaper cw THz equipment by collecting masking patterns in multiple planes and using numerical approaches such as the Gershberg–Saxton phase retrieval algorithm [70].

By the direct connection to the success of computational and compressed ghost imaging, future directions of this work could also include the numerical plane sectioning of target objects.
Finally, an important step for the application of speckle pattern modulation to real imaging systems would be the demonstration of successful imaging while using a continuously moving modulator. Modulator translation speeds necessary for real time video imaging would be easily achievable assuming bucket detector speed and reconstruction power were not limited. As a rough example, consider an extension of the active mode experiment at 16 by 16 resolution producing images at 30 Hz with 30% sampling. Each frame would require approximately 80 bucket measurements, each of which could be realized with a modulator translation of 0.5 mm (compare figure 4.6 on page 61). This would require a modulator translation speed of 120 cm/sec, which if realized with a spinning modulator disk of 25 cm (inside) diameter, would require a rotation speed of only 1.5 revolutions/sec. Such a spinning disk system in the transmission mode would constitute a direct extension of previous work [66, 67]. Moreover, the modulator translation speed required to achieve real time imaging could be further reduced by using with appropriate reconstruction methods suited for video [55, 69].

5.4 Conclusion

The experiments presented in this thesis have demonstrated the feasibility of a cheap and flexible modulation scheme for both active and passive mode compressed single pixel imaging based on objective speckle patterns for THz radiation. The use of highly random speckle patterns for illumination in the active mode arrangement can be considered as a direct extension of the technique of computational ghost imaging for THz, implemented with directly measured masking patterns in the near field. In addition, a unique extension of the method to the passive mode has been proposed and demonstrated for the first time. A relevant application of the speckle pattern modulation technique to passive millimeter wave imaging (PMMWI) systems has been identified, which may impact the system cost, complexity, and acquisition speed.
Part III

Appendices
Appendix A

The Discrete Cosine Transform

This section is dedicated to a pedagogical development and discussion of the two-dimensional discrete cosine transform (DCT), which was used to implement one of the two compressive sensing based reconstruction algorithms used in this work. By operating with sinusoid-like basis vectors, the DCT retains many compression properties of the discrete Fourier transform, without the need to involve complex numbers in the representation of real-valued signals and images.

For the purposes of this section and the extent to which we shall be concerned with vector spaces, we will always use the field of the real numbers $\mathbb{R}$ and regard vectors $\mathbf{a}$ as signals of length $N$ in $\mathbb{R}^N$. The coordinates of vectors in the standard basis $\{\mathbf{e}_i\}$ of $\mathbb{R}^N$ will be denoted as $x^{(i)}$, and thus $\mathbf{x} = \sum_{i=1}^{N} x^{(i)} \mathbf{e}_i$. We will also use the standard inner product: $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{i=1}^{N} a^{(i)} b^{(i)}$. 
A.1 Derivation of the 2D Transform Equation

**Theorem 1.** Suppose that the vectors $u_1, \ldots, u_N$ form an orthobasis of a vector space $V$ over a field $K$. Then every vector $x$ in $V$ can be uniquely expressed as a linear combination:

$$x = \sum_{j=1}^{N} \frac{\langle x, u_j \rangle}{\langle u_j, u_j \rangle} u_j.$$ 

Recognizing the fact that $\langle u_j, u_j \rangle = ||u_j||^2$ for the usual norm induced by the inner product, we may write the result of Theorem 1 as follows for the standard coordinates of a vector $x$:

$$x^{(i)} = \sum_{j=1}^{N} \frac{1}{||u_j||} \left( \frac{\langle x, u_j \rangle}{||u_j||} \right) u_j^{(i)}. \tag{A.1}$$

Moreover, since the inner products $\langle x, u_j \rangle$ in the case of vectors in $\mathbb{R}^N$ are computed (relative to the standard basis) as a sum by definition, we may define

$$X^{(j)} \equiv \frac{\langle x, u_j \rangle}{||u_j||} = \sum_{i=1}^{N} \frac{1}{||u_j||} x^{(i)} u_j^{(i)}. \tag{A.2}$$

Equations (A.1) and (A.2) in essence describe a ‘transform identity’ between the equally sized arrays of numbers $[x^{(i)}]_{i=1}^{N}$ and $[X^{(j)}]_{j=1}^{N}$. We shall record this fact as an example of a discrete transform induced by a choice of orthobasis:
Theorem 2. Suppose the vectors $u_1, \ldots, u_N$ form an orthobasis of $\mathbb{R}^N$ and let $x$ be an arbitrary vector in $\mathbb{R}^N$ with coordinates $x^{(i)}$. Then the following identity holds:

$$x^{(i)} = \sum_{j=1}^{N} \frac{1}{\|u_j\|} X^{(j)} u_j^{(i)},$$

where the numbers $X^{(j)}$ are defined as:

$$X^{(j)} = \sum_{i=1}^{N} \frac{1}{\|u_j\|} x^{(i)} u_j^{(i)}.$$

The version of the theorem stated above adopts the ‘symmetric’ convention that the factor of $1/\|u_j\|^2$ is shared equally between expressions. The vectors $u_j/\|u_j\|$ in fact form an orthonormal basis for $\mathbb{R}^N$.

In computing the transforms of two dimensional images, it will be of interest to obtain a generalized result of Theorem 2 appropriate for two dimensional arrays, i.e. matrices, over $\mathbb{R}$. We shall consider such image matrices, consisting of $k$ rows and $l$ columns, as elements in $\mathbb{R}^{k \times l}$. Reflecting this view, we shall adopt the notation $A^{(a,b)}$, where $a = 1, \ldots, k$ and $b = 1, \ldots, l$, for the coordinates of the matrix $A$ in $\mathbb{R}^{k \times l}$ corresponding to the element found at the intersection of row $a$ and column $b$. This convention has the effect of giving the coordinates of the vectors in $\mathbb{R}^{k \times l}$ the same lexicographic ordering as is standard for matrices. With this notation clear, we have the following result.
Theorem 3. Suppose that the vectors \( \{u_i\}_{i=1}^k \) and \( \{v_i\}_{i=1}^l \) are orthobases of the vector spaces \( \mathbb{R}^k \) and \( \mathbb{R}^l \) respectively. Then the set of matrices \( \{\omega_{ij}\}_{i=1}^k \}_{j=1}^l \) defined by

\[
\omega_{ij}^{(a,b)} = u_i^{(a)} v_j^{(b)}
\]

forms an orthobasis of \( \mathbb{R}^{k \times l} \).

Proof. We observe that the standard inner product defined over \( \mathbb{R}^{k \times l} \) is

\[
\langle \omega_{ab}, \omega_{cd} \rangle \equiv \sum_{i=1}^k \sum_{j=1}^l \omega_{ab}^{(i,j)} \omega_{cd}^{(i,j)}.
\]

Applying the definition of the matrices \( \{\omega_{ij}\}_{i=1}^k \}_{j=1}^l \), we have:

\[
= \sum_{i=1}^k \sum_{j=1}^l u_i^{(a)} v_j^{(j)} u_c^{(i)} v_d^{(j)} = \left( \sum_{i=1}^k u_i^{(i)} u_c^{(i)} \right) \left( \sum_{j=1}^l v_d^{(j)} v_d^{(j)} \right).
\]

Now the braced factors \( i \) and \( ii \) are equal to \( \langle u_a, u_c \rangle \) and \( \langle v_b, v_d \rangle \) respectively (where each inner product is the one defined in the corresponding vector space). This implies that the product \( \langle \omega_{ab}, \omega_{cd} \rangle \) is zero whenever \( a \neq c \) or \( b \neq d \), and moreover that when \( a = c \) and \( b = d \), \( \langle \omega_{ab}, \omega_{ab} \rangle = \|\omega_{ab}\|^2 = \|u_a\|^2 \|v_b\|^2 > 0 \) by the above equation.

With this proof in hand, we may now state a version of Theorem 2 which is applicable to our use of discrete transforms involving two-dimensional images:
**Theorem 4.** Let $\mathbb{R}^{k \times l}$ be the vector space of $k \times l$ real valued matrices and suppose that $\{u_i\}_{i=1}^k$ and $\{v_j\}_{j=1}^l$ are orthobases of $\mathbb{R}^k$ and $\mathbb{R}^l$ respectively. Then the following identity holds:

$$x^{(a,b)} = \sum_{i=1}^k \sum_{j=1}^l \frac{1}{\|u_i\| \|v_j\|} X^{(i,j)} u_i^{(a)} v_j^{(b)},$$

where the numbers $X^{(i,j)}$ are defined as:

$$X^{(i,j)} \equiv \sum_{a=1}^k \sum_{b=1}^l \frac{1}{\|u_i\| \|v_j\|} x^{(a,b)} u_i^{(a)} v_j^{(b)}.$$

It is worth noting that further generalizations of Theorem 4 are possible which extend the result up to an arbitrary number of sets of orthobases and thus an arbitrary number of dimensions. Versions of Theorems 2 and 4 can also be stated for the complex numbers which can be derived similarly to the above with the use of the standard inner product on $\mathbb{C}^N$: $\langle a, b \rangle = \sum_i a \overline{b_i}$.

We now adopt our particular choice of orthobasis on $\mathbb{R}^N$ for the purposes of Theorem 4:
Theorem 5. Let \( \{v_a\} \) be a set of vectors in \( \mathbb{R}^N \) defined by:

\[
v_a^{(x)} \equiv \cos \left( \frac{\pi}{N} \left( x + \frac{1}{2} \right) a \right),
\]

where \( x = 0, \ldots, N - 1 \) and \( a = 0, \ldots, N - 1 \). Then \( \{v_a\} \) forms an orthogonal basis of \( \mathbb{R}^N \) called the **Discrete Cosine Basis**. Moreover,

\[
||v_a|| = \begin{cases} \sqrt{N/2} & \text{if } a \neq 0 \\ \sqrt{N} & \text{if } a = 0 \end{cases}.
\]

(A.3)

**Proof.** We may express the cosine function in terms of complex exponentials as follows (writing the imaginary unit as \( i \)):

\[
v_a^{(x)} = \frac{1}{2} \left[ e^{i\pi a(x+1/2)/N} + e^{-i\pi a(x+1/2)/N} \right].
\]

(A.4)

The standard inner product on \( \mathbb{R}^N \) is now evaluated as:

\[
\langle v_a, v_b \rangle = \sum_{x=0}^{N-1} \frac{1}{2} \left[ e^{i\pi a(x+1/2)/N} + e^{-i\pi a(x+1/2)/N} \right] \times \frac{1}{2} \left[ e^{i\pi b(x+1/2)/N} + e^{-i\pi b(x+1/2)/N} \right].
\]

(A.5)
After collecting terms and simplifying, one obtains

\[
\langle \mathbf{v}_a, \mathbf{v}_b \rangle = \frac{1}{4} e^{i\pi(a+b)/2N} \sum_{x=0}^{N-1} (e^{i\pi(a+b)/N})^x \\
+ \frac{1}{4} e^{i\pi(a-b)/2N} \sum_{x=0}^{N-1} (e^{i\pi(a-b)/N})^x \\
+ \frac{1}{4} e^{i\pi(-a+b)/2N} \sum_{x=0}^{N-1} (e^{i\pi(-a+b)/N})^x \\
+ \frac{1}{4} e^{i\pi(-a-b)/2N} \sum_{x=0}^{N-1} (e^{i\pi(-a-b)/N})^x.
\]

(A.6)

Suppose that \(a \neq b\). It follows from the range of possible values for \(a\) and \(b\) that all of the sums \(\pm(a+b)\) and \(\pm(a-b)\) are non-zero and strictly bounded above and below by \(\pm|2N|\). These facts imply that none of the exponential terms involved as powers of \(x\) in the finite geometric sums above are equal to 1. Recalling that \(\sum_{x=0}^{N-1} r^x = (1 - r^N)/(1 - r)\) holds for complex \(r\) satisfying \(r \neq 1\), we have

\[
\langle \mathbf{v}_a, \mathbf{v}_b \rangle = \frac{1}{4} e^{i\pi(a+b)/2N} \begin{bmatrix} 1 - e^{i\pi(a+b)} \\ 1 - e^{i\pi(a+b)/N} \end{bmatrix} \\
+ \frac{1}{4} e^{i\pi(a-b)/2N} \begin{bmatrix} 1 - e^{i\pi(a-b)} \\ 1 - e^{i\pi(a-b)/N} \end{bmatrix} \\
+ \frac{1}{4} e^{i\pi(-a+b)/2N} \begin{bmatrix} 1 - e^{i\pi(-a+b)} \\ 1 - e^{i\pi(-a+b)/N} \end{bmatrix} \\
+ \frac{1}{4} e^{i\pi(-a-b)/2N} \begin{bmatrix} 1 - e^{i\pi(-a-b)} \\ 1 - e^{i\pi(-a-b)/N} \end{bmatrix}.
\]

(A.7)

Simplifying further, and using the identity \(e^{iz}/(1 - e^{iz}) = -1/(2 \sin(z/2))\) yields

\[
\langle \mathbf{v}_a, \mathbf{v}_b \rangle = \frac{1}{8} \sin(\pi(a+b)/2N) \frac{(-1)^{a+b} - 1}{\sin(\pi(a+b)/2N)} \\
+ \frac{1}{8} \sin(\pi(a-b)/2N) \frac{(-1)^{a-b} - 1}{\sin(\pi(a-b)/2N)} \\
+ \frac{1}{8} \sin(\pi(-a+b)/2N) \frac{(-1)^{-a+b} - 1}{\sin(\pi(-a+b)/2N)} \\
+ \frac{1}{8} \sin(\pi(-a-b)/2N) \frac{(-1)^{-a-b} - 1}{\sin(\pi(-a-b)/2N)}.
\]

(A.8)

which is equal to 0 due to the cancellation of terms after applying \((-1)^{-y} = (-1)^y\)
and the odd symmetry of the sine function. This demonstrates that $\langle v_a, v_b \rangle = 0$ whenever $a \neq b$.

Evaluating $\langle v_a, v_a \rangle$ involves many of the same simplifications as used above which we further use without comment. Starting from equation (A.6), we have

$$\langle v_a, v_a \rangle = \frac{1}{4} e^{i \pi a/N} \sum_{x=0}^{N-1} e^{i \pi 2a/N} x + \frac{N}{4} + \frac{N}{4}$$

$$+ \frac{1}{4} e^{i \pi (-a)/N} \sum_{x=0}^{N-1} e^{i \pi (-2a)/N} x. \quad (A.9)$$

If we have $a \neq 0$, then the ratio $a/N$ is never an integer and thus we may simplify using the finite geometric series formula etc. as before to obtain

$$\langle v_a, v_a \rangle = \frac{N}{2} + \frac{1}{8} \left( (-1)^{2a} - 1 \right) + \frac{1}{8} \left( (-1)^{-2a} - 1 \right)$$

$$= \frac{N}{2} \quad (A.10)$$

due to the zero numerators. Finally, if $a = 0$, then (A.6) simply becomes

$$\langle v_a, v_a \rangle = \frac{N}{4} + \frac{N}{4} + \frac{N}{4} + \frac{N}{4} = N. \quad (A.11)$$

It should be noted that multiple non-identical definitions of the discrete cosine transform exist in the signal processing literature. The transform adopted in theorem 5 for the purposes of this work is the ‘DCT-II’ transform, as classified by Rao and Yip [77].

Theorems 4 and 5 now furnish a presentation of the 2-dimensional discrete cosine transform [78].
Theorem 6. Let $\mathbb{R}^{k \times l}$ be the vector space of $k \times l$ real valued matrices whose (row, column) indices are indexed starting from zero. Then the following identity holds:

$$x^{(a,b)} = \frac{1}{\sqrt{kl}} \sum_{i=0}^{k-1} \sum_{j=0}^{l-1} X^{(i,j)} \alpha(i) \alpha(j) \cos \left( \frac{\pi}{k} (a + 1/2)i \right) \cos \left( \frac{\pi}{l} (b + 1/2)j \right),$$

where the numbers $X^{(i,j)}$ are defined as:

$$X^{(i,j)} \equiv \frac{1}{\sqrt{kl}} \sum_{a=0}^{k-1} \sum_{b=0}^{l-1} x^{(a,b)} \alpha(i) \alpha(j) \cos \left( \frac{\pi}{k} (a + 1/2)i \right) \cos \left( \frac{\pi}{l} (b + 1/2)j \right),$$

and the functions $\alpha(y)$ are defined as

$$\alpha(y) \equiv \begin{cases} 
1 & \text{if } y = 0 \\
\sqrt{2} & \text{if } y \neq 0
\end{cases}.$$

The 2D basis matrices used in Theorem 6 are visualized in Figure A.1 for the case $k = l = 8$. The basis matrix $\omega^{(a,b)}_{ij} \equiv u_i^{(a)} v_j^{(b)}$ (where $i, j = 0, \ldots, 7$) is shown in the $(i+1, j+1)$ position in the figure. The matrix shown in the top left $(1,1)$ corresponds to a constant-valued matrix whose coefficient under the DCT measure the average pixel value in the transformed image.
Figure A.1: A visualization of the 64 orthogonal basis matrices used by the 2D discrete cosine transform over the space of $8 \times 8$ matrices. Positive values are shown in white, and negative values are shown in black.
A.2 Sparsity of Real World Images

We now introduce the concept of a lossy compression scheme based on the above transformations. For a certain class of signals \( x^{(i)} \), it is often possible to adopt a choice of orthobasis in \( \mathbb{R}^N \) for which the transformed signals \( X^{(j)} \) have a small number of ‘large’ coordinates (in magnitude). If the ‘small’ magnitude coordinates are discarded (that is set identically to zero) to obtain a signal \( \tilde{X}^{(j)} \), then a compressed signal \( \tilde{x}^{(i)} \) approximating \( x^{(i)} \) is obtained from the inverse transformation of \( \tilde{X}^{(j)} \). That is, the signal \( x^{(i)} \) can be represented through the transformation by the signal \( \tilde{X}^{(j)} \) which can be stored in memory at a reduced size due to the discarded coordinates. Such a compression scheme is often referred to as **transform encoding**.

The DCT defines such a compression scheme for the class of ‘real world’ images which are often encountered in image sensing. Consider the discrete cosine transformation used in figure A.2. A grayscale 2500 pixel image of the planet Saturn (a) and its coefficients under the DCT (b) are both shown. What is of interest is the fact that the largest DCT coefficients are all collected into the upper left hand corner, corresponding to the ‘lower frequency’ components of the image. A reduction in the number of coefficients needed to represent the image is then implemented by only selecting the largest 10\% of coefficients based on their absolute values. The locations of these coefficients in (b) is shown in (d) using white pixels. This again highlights the tendency of large coefficients in the Saturn image to collect in the upper left hand corner of the DCT image matrix.

We may observe in general that images with primarily low-frequency content have few large coefficients under the DCT. High-frequency content in images often correspond to features which appear like random noise. For imaging applications involving real objects with smoothly-behaved reflectance functions, the DCT and other transform schemes offer a tractable encoding scheme for lossy compression. The orig-
Figure A.2: A demonstration of transform compression using the DCT. A $50 \times 50$ grayscale image of the planet Saturn is shown in (a). The array of DCT coefficients is visualized as an image matrix in (b), with the DC coefficient occupying the top left hand corner. The 10% of all coefficients with the largest absolute value are highlighted in white in (d). Selecting only these largest coefficients while setting the rest identically equal to zero results in the compressed image of Saturn (c).

inal implementation of the JPEG image encoding scheme utilized the DCT on $8 \times 8$ pixel sub-blocks of the image across each color channel. These small blocks represented portions of the image over which the reflection data could be assumed to be smoothly-behaved with low high-frequency content.
APPENDIX A. THE DISCRETE COSINE TRANSFORM

Figure A.3: Several transform compressed 50 × 50 images of Saturn using the same procedure as that used in Figure A.2. The displayed percentages indicate the proportion of coefficients selected for reconstruction based on their order of largest absolute value.
Appendix B

Estimation of $\epsilon$ for Image Reconstruction

Chapter 2 was concerned with the development of the following two noise-compatible reconstruction algorithms used for this thesis:

$$(\text{minDCT}) \quad \hat{z} = \arg \min \|z\|_1 \quad \text{subject to} \quad \|\Phi \Psi z - y\|_2 \leq \epsilon$$

$$(\text{minTV}) \quad \hat{x} = \arg \min \quad \text{TV} (x) \quad \text{subject to} \quad \|\Phi x - y\|_2 \leq \epsilon$$

where $\Phi$ was the $M$ by $N$ measurement matrix with observations $y$, and $\Psi$ was the $N$ by $N$ matrix which induced the discrete cosine transform (DCT). Both reconstruction algorithms depend on a single error tolerance parameter, $\epsilon$, which strongly determines the reconstruction quality. In this section, a method of estimating $\epsilon$ is derived from basic assumptions regarding the noise properties of the single pixel imaging system. As a result, the problem of choosing an appropriate value of $\epsilon$ from $[0, \infty)$, reduces to a tunable selection from $[0, 1]$. Another feature of the statistical model presented here is the incorporation of a normalization parameter which sets the values of the reconstructed imaging results into or near the physically required range of $[0, 1]$. 

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B.1 Statistical Model

We summarize our noise model for the observation vector \( y \) as follows:

\[
y = \alpha b = (\Phi + Z)x + e,
\]  

where the vectors \( x \in \mathbb{R}^N \), vectors \( y \) and \( b \in \mathbb{R}^M \) and the matrix \( \Phi \in \mathbb{R}^{M \times N} \) are respectively the underlying transmission vector, the algorithm input observation vector, the experimental bucket signal vector, and the measurement matrix. Equation (B.1) thus expresses both the normalized input signal \( y \) and the observed bucket signals \( b \) in terms of both \( x \) and \( \Phi \) in addition to the parameter \( \alpha \), and the additive noise components \( Z \) and \( e \).

The imaging experiments presented in this thesis consist of two steps: mask extraction, and bucket signal collection. In principal these two stages may be carried out using different photodetectors with possibly different responsivities and noise characteristics. The positive scalar \( \alpha \) acts as an amplification factor accounting for signal strength discrepancies between mask extraction and image sampling. A correctly compensating choice of \( \alpha \) allows for the reconstruction algorithms to reliably produce well-scaled transmission or reflection functions \( x \) where each pixel \( x_i \) is found in the physical range \([0, 1]\). A method of choosing such an \( \alpha \) will be presented shortly.

The experimental noise floor of the bucket signal detector is represented by the additive noise vector \( e \) whose elements \( e_i \) are each i.i.d. normal with a mean \( \mu \) and variance \( \sigma^2_e \). In practice, \( \mu > 0 \) is observed due to spurious intensity signals even from a completely covered detector. In addition, additive noise affecting the masking patterns was modeled by the \( M \) by \( N \) noise matrix \( Z \) with i.i.d. normal elements with zero mean and variance \( \sigma^2_Z \). In practice, it was observed that appropriate values of \( \epsilon \) for reconstruction varied depending on the underlying transmission image. This
fact is captured in the noise model by $Z$ whose contribution to the normalized bucket signals $y$ is determined through the matrix-vector product $Zx$.

### B.1.1 Parameter Extraction Procedure

The noise model stated in equation (B.1) incorporates three components, $\alpha$, $Z$ and $e$, which ultimately characterize experimental noise through the noise parameters $\mu$, $\sigma_e$ and $\sigma_Z$. For any given speckle pattern imaging arrangement, all of these statistics may be extracted taking calibration measurements of a completely open and a completely blocked imaging plane, corresponding to underlying transmission function vectors $x_{\text{open}}$ and $x_{\text{closed}}$ whose elements are each identically one and zero, respectively. In what follows, we shall write $b_{\text{open}}$ and $b_{\text{closed}}$ to represent the experimental bucket signal vectors taken of the transparent and blocked imaging planes.

Before $\mu$, $\sigma_e$ and $\sigma_Z$ can be extracted, an appropriate amplification factor $\alpha$ must be chosen. From our noise model (B.1), we may write for the open transmission image samples:

$$\alpha b_{\text{open}} = \Phi x_{\text{open}} + Zx_{\text{open}} + e$$

$$= \Sigma + Z + e \quad \text{(B.2)}$$

where $\Sigma$ and $Z$ each represent $M$-dimensional column vectors equal to the element-wise sum across the rows of $\Phi$ and $Z$. Now averaging across the $M$ vector elements, we obtain the scalar equation

$$\alpha \langle b_{\text{open}} \rangle = \langle \Sigma \rangle + \mu \quad \text{(B.4)}$$

where we have written $\mu = \langle e \rangle$ via the mean value estimator and observed that $\langle Z \rangle = 0.$
APPENDIX B. ESTIMATION OF $\epsilon$ FOR IMAGE RECONSTRUCTION

For our closed transmission image samples, we may similarly write

$$\alpha b_{\text{closed}} = \Phi x_{\text{closed}} + Z x_{\text{closed}} + \epsilon \quad (B.5)$$

$$= \epsilon \quad (B.6)$$

from which we may obtain

$$\alpha \langle b_{\text{closed}} \rangle = \mu \quad . \quad (B.7)$$

After eliminating $\mu$, equations (B.4) and (B.7) may be solved for $\alpha$. The result is:

$$\alpha = \frac{\langle \Sigma \rangle}{\langle b_{\text{open}} \rangle - \langle b_{\text{closed}} \rangle} \quad . \quad (B.8)$$

Once $\alpha$ is chosen, the same calibration data $b_{\text{open}}$ and $b_{\text{closed}}$ may be used to estimate $\sigma_\epsilon$ and $\sigma_Z$. To see this, we may rearrange equations (B.2) and (B.5) as follows:

$$\alpha b_{\text{open}} - \Sigma = Z + \epsilon \quad (B.9)$$

$$\alpha b_{\text{closed}} = \epsilon \quad (B.10)$$

Histogram data taken across the $M$-vectors $\alpha b_{\text{open}} - \Sigma$ and $\alpha b_{\text{closed}}$ may therefore be used to indirectly estimate $\mu$, $\sigma_\epsilon$ and $\sigma_Z$. In particular, we note that both $\mu$ and $\sigma_\epsilon$ may be estimated directly since:

$$(\alpha b_{\text{closed}})_i \sim N \left( \mu, \sigma_\epsilon^2 \right) \quad (B.11)$$

where we have written the subscript $i$ to indicate the i.i.d. vector elements. As for $\alpha b_{\text{open}} - \Sigma$, since its distribution is determined as a linear combination of normal random variables, we can conclude that it must also be normally distributed according
APPENDIX B. ESTIMATION OF $\epsilon$ FOR IMAGE RECONSTRUCTION

to:

$$(\alpha b_{\text{open}} - \mathbf{\Sigma})_i \sim N(\mu, A^2)$$

(B.12)

where we have set $A^2 \equiv N \sigma_Z^2 + \sigma_e^2$. Thus our final statistic may be extracted:

$$\sigma_Z^2 = \frac{A^2 - \sigma_e^2}{N}.$$  \hspace{1cm} (B.13)

**B.2 Estimating $\epsilon$**

In both of the minDCT and minTV reconstruction algorithms, the constraint representing acceptable agreement with the observed bucket signals reads as the inequality:

$$||y - \Phi x||_2 \leq \epsilon $$ \hspace{1cm} (B.14)

where as before $y$ represents the input bucket signals and $x$ represents a candidate transmission or reflection function across the imaging plane. Now incorporating our modeling of bucket signal noise from equation (B.1), we may rewrite the above equation as follows:

$$||y - \Phi x||_2 = ||\alpha b - \Phi x||_2 = ||Zx + e||_2.$$ \hspace{1cm} (B.15)

According to the statistical properties of the noise matrix $Z$ and vector $e$, we cannot in general expect the quantity $||Zx + e||_2$ to be as small as we like in practice. Appropriate values of epsilon are therefore those values which with a high likelihood serve as upper bounds for the $\ell_2$ norm of the error residual.

Recalling that the elements of the matrix $Z$ are i.i.d. normal with zero mean and variance $\sigma_Z^2$, we observe that the matrix vector product $Zx$ forms a linear combination of i.i.d. normal random variables with coefficients determined by the image vector $x$. 
APPENDIX B. ESTIMATION OF $\epsilon$ FOR IMAGE RECONSTRUCTION

In particular for elements of the vector $Z\mathbf{x}$ we may write:

$$(Z\mathbf{x})_i = \sum_{j=1}^{N} z_{ij} x_j$$  \hspace{1cm} (B.16)

and therefore conclude the following distribution of multiplication elements:

$$(Z\mathbf{x})_i \sim N \left(0, \left[ \sum_{j=1}^{N} x_j^2 \right] \sigma_Z^2 \right) = N \left(0, ||\mathbf{x}||_2^2 \sigma_Z^2 \right)$$  \hspace{1cm} (B.17)

With this result it now follows that

$$(Z\mathbf{x} + e)_i \sim N \left(\mu, ||\mathbf{x}||_2^2 \sigma_Z^2 + \sigma_e^2 \right)$$  \hspace{1cm} (B.18)

and thus

$$\frac{(Z\mathbf{x} + e)_i - \mu}{\sqrt{||\mathbf{x}||_2^2 \sigma_Z^2 + \sigma_e^2}} \sim N(0,1)$$  \hspace{1cm} (B.19)

We may now find the distribution of the statistic defined as the sum over the squares of each element of the $M$-vector $Z\mathbf{x} + e$ in equation (B.19) as follows:

$$\sum_{i=1}^{M} \left( \frac{(Z\mathbf{x} + e)_i - \mu}{\sqrt{||\mathbf{x}||_2^2 \sigma_Z^2 + \sigma_e^2}} \right)^2 = \frac{||Z\mathbf{x} + e - \mu\mathbf{I}_M||_2^2}{||\mathbf{x}||_2^2 \sigma_Z^2 + \sigma_e^2} \sim \chi_M^2$$  \hspace{1cm} (B.20)

where $\mathbf{I}_M$ represents the $M$-vector with all elements equal to one, and the function $\chi_M^2$ represents the Chi-squared distribution with $M$ degrees of freedom. The known mean and variance of the $\chi_M^2$ distribution (which are given as $M$ and $2M$ respectively) now allow us to form an upper bound from the confidence interval of $\lambda$ standard deviations:

$$\frac{||Z\mathbf{x} + e - \mu\mathbf{I}_M||_2^2}{||\mathbf{x}||_2^2 \sigma_Z^2 + \sigma_e^2} < M + \lambda \sqrt{2M}$$  \hspace{1cm} (B.21)
APPENDIX B. ESTIMATION OF $\epsilon$ FOR IMAGE RECONSTRUCTION

Rearranging equation (B.21) results in

$$||Zx + e - \mu I_M||_2 < \sqrt{||x||_2^2 \sigma_Z^2 + \sigma_e^2 \sqrt{M + \lambda \sqrt{2M}}}$$

(B.22)

from which the triangle inequality for the $\ell_2$ norm\textsuperscript{1} now implies

$$||Zx + e||_2 - ||\mu I_M||_2 < \sqrt{||x||_2^2 \sigma_Z^2 + \sigma_e^2 \sqrt{M + \lambda \sqrt{2M}}}.$$  \hspace{1cm} (B.23)

Finally, evaluating $||\mu I_M||_2$ as $\mu \sqrt{M}$, we may write:

$$||Zx + e||_2 < \mu \sqrt{M} + \sqrt{||x||_2^2 \sigma_Z^2 + \sigma_e^2 \sqrt{M + \lambda \sqrt{2M}}}.$$  \hspace{1cm} (B.24)

The right hand side of equation (B.24) now provides a statistical upper bound for $\epsilon$ of confidence strength $\lambda$. Unfortunately, the strong dependence of reconstruction quality on $\epsilon$ means that a large upper bound does not produces satisfactory reconstruction results in all cases. Therefore, to provide for a variable adjustment of $\epsilon$ within its upper bounded range, a reconstruction tuning factor $\omega$ is therefore introduced with values in the range $[0, 1]$: \hspace{1cm} (B.25)

$$||Zx + e||_2 < \omega \left( \mu \sqrt{M} + \sqrt{||x||_2^2 \sigma_Z^2 + \sigma_e^2 \sqrt{M + \lambda \sqrt{2M}}} \right) .$$

Equation (B.25) now provides a tunable estimate for the reconstruction error residual based on the tuning factor $\omega$, and experimental parameters $M$, $\mu$, $\sigma_e$, and $\sigma_Z$, in addition to the underlying image norm $||x||_2$. The value of the upper bound confidence used in all cases was $\lambda = 1.0.$

\textsuperscript{1}We here use the triangle inequality in the equivalent form $||x||_2 - ||y||_2 \leq ||x - y||_2.$
B.2.1 Estimating $\|x\|_2$

As shown above, a reasonable estimate for the error parameter $\epsilon$ requires an estimation of $\|x\|_2$, the euclidean norm of the ‘ideal’ or ‘actual’ transmission function present in the imaging plane. Although additional a priori assumptions about the transmission function may furnish an estimate for this quantity in some circumstances, an iterative method may be presented as means of obtaining image reconstructions without such assumptions.

We may choose an initial image norm estimate of $\|x\|_2 \approx \sqrt{N/2}$, representing an initial estimate of the image norm for a fully binary image with 50% transparent and 50% opaque pixel values. Solving either reconstruction problem with $\epsilon$ chosen according to equation (B.25) now produces an associated image reconstruction vector $x_1$. Albeit suffering from a potentially poor estimate of error parameter, $x_1$ now allows us to furnish a new epsilon estimate $\epsilon_2$ using its norm, and in general justifies the following iterations based on the statistical upper bounds:

$$\epsilon_{n+1} = \omega \left( \mu \sqrt{M} + \sqrt{\|x_n\|_2^2 \sigma^2 \sigma^2 Z + \sigma^2 \sqrt{M + \lambda \sqrt{2M}}} \right), \quad (B.26)$$

where $x_n$ represents the reconstructed image using the previous error estimate $\epsilon_n$.

By iterating a series of $\epsilon$ estimates in this way, a sequence of image reconstructions $x_n$ may be obtained in a semi-adaptive manner. In practice, fast convergence of the $\epsilon_n$ parameters was observed after only a few iterations for most experiments. In all cases presented in this thesis, three iterations were used to produce reconstructions.

B.3 Experimental Results

The above program of data ingestion and parameter estimation was followed for all reconstruction results across both active and passive mode imaging experiments. In
the course of the active mode experiment, a full set of $M = 256$ masks was used for the calculation of the noise statistics $\mu$, $\sigma_e$, and $\sigma_Z$. Histograms used to produce these active mode estimations are shown in figure B.1. For the smaller resolution line-scans produced in the passive mode experiment, smaller sets of $M = 20$ masks were used to produce estimates of the same parameters. Histograms for the passive mode experiment are shown in figure B.2, demonstrating the much larger impact of noise in the measurements as compared to the active mode experiment. The noise parameter values extracted from the histograms were held constant across all $\epsilon$ estimations for each experiment, respectively. In all cases, the bell shape of these histograms supports the assumptions of the i.i.d. normal additive noise components. Moreover, each pair of histograms shares an identical mean as a result of the choice of the normalization factor $\alpha$. 
Figure B.1: Histogram results shown for the $M = 256$ dimensional vectors $\alpha b_{\text{closed}}$ and $\alpha b_{\text{open}} - \Sigma$, used to produce estimations of $\mu$, $\sigma_\epsilon$, and $\sigma_Z$ in the active mode imaging experiment. The resulting parameters used were: $\alpha = 1.08$, $\mu = 0.888$, $\sigma_Z = 0.0780$, and $\sigma_\epsilon = 0.0853$. 
Figure B.2: Histogram results shown for the $M = 20$ dimensional vectors $\alpha b_{\text{closed}}$ and $\alpha b_{\text{open}} - \Sigma$, used to produce estimations of $\mu$, $\sigma_e$, and $\sigma_Z$ in the passive mode imaging experiment. The resulting parameters used were: $\alpha = 30.3$, $\mu = 7.88$, $\sigma_Z = 21.3$, and $\sigma_e = 2.91$. 
Appendix C

Gallery of Reconstruction Results
### C.1 Active Mode Imaging Results

Figure C.1: Legend for the interpretation of the gallery plates of the active mode imaging results. Facsimiles of the seven binary target masks are shown at left, with the completely closed and open calibration images shown in the bottom two rows respectively. The percentages along the top of each column correspond to the sampling ratios compared to the total number of pixels in each image ($N = 256$). The numbers inside of each square show the tuning factors used across each reconstruction for that position.

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Figure C.2: Gallery of minDCT reconstruction results with averaging 1.
Figure C.3: Gallery of minTV reconstruction results with averaging 1.
Figure C.4: Gallery of minDCT reconstruction results with averaging 10.
Figure C.5: Gallery of minTV reconstruction results with averaging 10.
## C.2 Passive Mode Imaging Results

![Table and Diagram]

Figure C.6: Legend for the interpretation of the gallery plates of the passive mode imaging results. The target patterns (i)-(iv) are illustrated at bottom with patterns (iii) and (iv) corresponding to the completely closed and open calibration images respectively. The percentages along the top of each column correspond to the sampling ratios as percentages of the pixels in each line scan image \((N = 20)\). The numbers inside of each square show the tuning factors used across each reconstruction for that position.
Figure C.7: Passive minDCT reconstructions with averaging 1. The asterisk symbol (*) indicates complete overlap of both curves.

Figure C.8: Passive minTV reconstructions with averaging 1.
Figure C.9: Passive minDCT reconstructions with averaging 10. The asterisk symbol (*) indicates complete overlap of both curves.

Figure C.10: Passive minTV reconstructions with averaging 10.
Figure C.11: Passive minDCT reconstructions with averaging 50. The asterisk symbol (*) indicates complete overlap of both curves.

Figure C.12: Passive minTV reconstructions with averaging 50.
Bibliography


